

Distance between unitary orbits of normal elements in simple C^* -algebras of real rank zero

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Abstract

Let x, y be two normal elements in a unital simple C^* -algebra A . We introduce a function $D_c(x, y)$ and show that in a unital simple AF-algebra there is a constant $1 > C > 0$ such that

$$C \cdot D_c(x, y) \leq \text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq D_c(x, y),$$

where $\mathcal{U}(x)$ and $\mathcal{U}(y)$ are the closures of the unitary orbits of x and of y , respectively. We also generalize this to unital simple C^* -algebras with real rank zero, stable rank one and weakly unperforated K_0 -group. More complicated estimates are given in the presence of non-trivial K_1 -information.

1 Introduction

Let H be a Hilbert space and let $B(H)$ be the C^* -algebra of all bounded operators. The study of normal operators in $B(H)$ has a long history. It has been an interesting and important problem to determine when two normal operators are unitarily equivalent in a subalgebra A of $B(H)$. Any detailed account of history will inevitably involve an enormous amount of literature. We will choose to limit ourselves to the immediate concerns of this paper. We study the distance between unitary orbits of normal elements. Some of the pioneer works on this subject have been made by Ken Davidson in [6], [5] and [7]. More recent work on the subject can be found in [29] and [30].

Let A be a unital C^* -algebra and let $x \in A$ be a normal element. The unitary orbit of x is defined to be the set $\{u^*xu : u \in U(A)\}$, where $U(A)$ is the unitary group of A . Denote by $\mathcal{U}(x)$ the closure of the unitary orbit of x . Suppose that $y \in A$ is another normal element. Denote by X and Y the spectrum of x and of y , respectively. Let $\varphi_X, \varphi_Y : C(X \cup Y) \rightarrow A$ be the homomorphisms defined by $\varphi_X(f) = f(x)$ and $\varphi_Y(f) = f(y)$ for all $f \in C(X \cup Y)$. Suppose that A has a unique tracial state τ . Denote by $\mu_{\tau \circ \varphi_X}$ and $\mu_{\tau \circ \varphi_Y}$ the two probability measures on $X \cup Y$ defined by the positive linear functionals $\tau \circ \varphi_X$ and $\tau \circ \varphi_Y$, respectively. For each open subset $O \subset \mathbb{C}$ and $r > 0$, denote by $O_r = \{\xi \in \mathbb{C} : \text{dist}(\xi, O) < r\}$. Define

$$r_O = \inf\{r : \mu_{\tau \circ \varphi_X}(O_r) \leq \mu_{\tau \circ \varphi_Y}(O_r)\}$$

and define

$$D_T(x, y) = \sup\{r_O : O \text{ open subset of } \mathbb{C}\}.$$

For finite dimensional Hilbert spaces, when $A = M_n$, the $n \times n$ matrix algebra over \mathbb{C} , an application of the Marriage Theorem shows that

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq D_T(x, y) \tag{e 1.1}$$

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(see [12] and [6], for example).

We first realize that (e 1.1) also holds for the case that A is a UHF-algebra. Apart from the application of the Marriage Theorem, the proof is also based on an important fact that normal elements in A can be approximated by normal elements with finite spectrum ([15]). If we allow A to be a general unital simple AF-algebra with a unique tracial state, the above bound no longer works because of the presence of possible infinitesimal elements in $K_0(A)$. Even without infinitesimal elements in $K_0(A)$, in the case that A has infinitely many extremal tracial states, the usage of the Marriage Theorem has to mix appropriately with the Riesz interpolation property. With the help of the Cuntz semigroups, which are more appropriate tools to compare positive elements, we are able to establish a modified upper bound formula for the distance between unitary orbits of two normal elements in a general unital AF-algebra (see 3.7 below). The fact that normal elements in an AF-algebra can be approximated by normal elements with finite spectrum plays an essential role in the proof. This follows from the result that a pair of almost commuting self-adjoint matrices is close to a pair of commuting self-adjoint matrices (see [16] and [11]). In [17], it was shown that, in a unital separable simple C^* -algebra of real rank zero, stable rank one and with weakly unperforated $K_0(A)$, a normal element x can be approximated by those normal elements with finite spectrum if $\lambda - x \in \text{Inv}_0(A)$ (the connected component of invertible elements of A containing the identity) for all $\lambda \notin \text{sp}(x)$. In this case we also prove that the same upper bound works for distance between unitary orbits of two normal elements in A which have vanishing K_1 information (see 3.6).

The distance between unitary orbits of normal elements in unital purely infinite simple C^* -algebras were recently studied in [29]. In that case, one could get a precise formula for the distance at least for the case that $K_0(A) = 0$ when no K_1 -information involved. However, the distance is basically given by $d_H(\text{sp}(x), \text{sp}(y))$, the Hausdorff distance between the spectra. It is the presence of the trace in the finite C^* -algebras that makes our upper bound more complicated and sophisticated. But it is exactly this phenomenon that is exciting. However, it is more desirable, in this current study, to include the cases that C^* -algebras have non-trivial K_1 -groups and normal elements have non-trivial K_1 -information.

To study the unitary orbits of normal elements, one has to know when two normal elements are approximately unitarily equivalent. When A is a unital purely infinite simple C^* -algebra, or A is a unital separable simple C^* -algebra with finite tracial rank, we know exactly when two normal elements are approximately unitarily equivalent (see [4], [21] and [23]). These works actually deal with the problem of unitary orbits of homomorphisms from $C(X)$, the C^* -algebra of continuous functions on a compact metric space X , to a unital purely infinite simple C^* -algebra or a unital separable simple C^* -algebra with finite tracial rank. These studies are closely related to the Elliott program of classification of amenable C^* -algebras.

To consider the unitary orbits of normal elements in a unital separable simple C^* -algebra A with real rank zero, stable rank one and weakly unperforated $K_0(A)$, we first present a theorem that two normal elements $x, y \in A$ are approximately unitarily equivalent if and only if $\text{sp}(x) = \text{sp}(y)$, $(\varphi_x)_*i = (\varphi_y)_*i$, $i = 0, 1$ and $\tau \circ \varphi_x = \tau \circ \varphi_y$ for all $\tau \in T(A)$ (the tracial state space of A), where $\varphi_x, \varphi_y : C(\text{sp}(x)) \rightarrow A$ are defined by $\varphi_x(f) = f(x)$ and by $\varphi_y(f) = f(y)$ for all $f \in C(\text{sp}(x))$, respectively. This is a generalization of the similar result in [21] (which only works for unital simple C^* -algebra with tracial rank zero).

Let A be a unital separable simple C^* -algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$ and let $x, y \in A$ be two normal elements with $\text{sp}(x) = X$ and $\text{sp}(y) = Y$, respectively. We first consider the case that $(\lambda - x)^{-1}(\lambda - y) \in \text{Inv}_0(A)$ for all $\lambda \notin X \cup Y$. With the help of a Mayer-Vietoris Theorem, we are able to present a reasonable upper bound for the distance between unitary orbits of x and y (see 6.7 below) in the same spirit of 3.6 mentioned above. However, there are normal elements with $\text{sp}(x) = \text{sp}(y)$ which induce exactly the same

map on the Cuntz semigroup, but, for any given $\lambda \notin X \cup Y$, $(\lambda - x)^{-1}(\lambda - y) \notin \text{Inv}_0(A)$. In this case the same upper bound mentioned above is zero. Nevertheless, as first found by Davidson ([5]),

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \geq \sup\{\text{dist}(\lambda, \text{sp}(x)) + \text{dist}(\lambda, \text{sp}(y))\}, \quad (\text{e 1.2})$$

where the supremum is taken among those $\lambda \notin \text{sp}(x) \cup \text{sp}(y)$ such that $(\lambda - x)^{-1}(\lambda - y) \notin \text{Inv}_0(A)$.

Based on 6.7, we also present an upper bound for distance between unitary orbits of normal elements in A , which is a combination of the upper bound in 6.7 together with the spirit of the lower bound given by Davidson (see 7.3 below).

Then, of course, there is the issue of the lower bound. It was shown by Davidson ([6]) that there is a universal constant $1 > C > 0$ such that

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \geq C \cdot D_T(x, y) \quad (\text{e 1.3})$$

in the case that $A = M_n$ (or $A = B(H)$ for infinite dimensional Hilbert space H with some modification). The constant is computed at least $1/3$. It was shown even in the case that $n \geq 3$, the constant C cannot be made equal to 1 ([13]). We show that a similar lower bound (with the same constant C) holds for unital AF-algebras and, more generally, for unital separable simple C^* -algebras of tracial rank zero. A different lower bound $d_c(x, y)$ is also presented. There are cases that $d_c(x, y) = D_c(x, y)$ with $\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) = d_c(x, y)$ and cases that $d_c(x, y) < D_c(x, y)$ with $\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) = D_c(x, y)$.

Briefly, the paper is organized as follows: Section 2 serves as a preliminary of the paper. Some metrics associated with the measure distribution and the Cuntz semigroups are introduced. In Section 3, we present an upper bound for the distance between unitary orbits of normal elements in unital AF-algebras and in unital separable simple C^* -algebras A with real rank zero, stable rank one and weakly unperforated $K_0(A)$, when K_1 -information of the relevant normal elements vanish. In Section 4, we show, with the metric introduced in Section 2, that normal elements can always be approximated by normal elements with finite spectrum in a unital simple C^* -algebra with stable rank one, real rank zero and weakly unperforated $K_0(A)$. Another function $D_c^e(x, y)$ which is a modification of $D_c(x, y)$ is also introduced. In Section 5, we show exactly when two normal elements in A are approximately unitarily equivalent. In Section 6, we present an upper bound for the distance between unitary orbits of normal elements $x, y \in A$ under the condition that $\lambda - x$ and $\lambda - y$ give the same K_1 element (but not necessarily zero) when $\lambda \notin \text{sp}(x) \cup \text{sp}(y)$. In Section 7, we present a general upper bound for the distance between unitary orbits of normal elements in A (without any assumption on K_1 -information). In Section 8, we discuss the lower bound of the distance between unitary orbits of normal elements.

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2 Preliminaries

Definition 2.1. Let A be a unital C^* -algebra. Denote by $U(A)$ the unitary group of A . Let $x \in A$ be a normal element. Define $\mathcal{U}(x)$ to be the closure of

$$\{u^*xu : u \in U(A)\}.$$

Definition 2.2. Fix a compact metric space Ω . Let $r > 0$. For each subset $S \subset \Omega$, define

$$S_r = \{t \in \Omega : \text{dist}(t, S) < r\}, \quad S_{-r} = \{t \in \Omega : \text{dist}(t, S^c) > r\}$$

and, denote by \bar{S} the closure of S .

Definition 2.3. Let A be a unital C^* -algebra. Denote by $T(A)$ the tracial state space of A . Let $\text{Aff}(T(A))$ be the space of all real affine continuous functions on $T(A)$. Denote by $\rho_A : K_0(A) \rightarrow \text{Aff}(T(A))$ the order preserving homomorphism defined by $\rho_A([p])(\tau) = (\tau \otimes \text{Tr})(p)$ for all projections in $M_n(A)$, where Tr is the standard trace on M_n , $n = 1, 2, \dots$

Definition 2.4. Let A be unital C^* -algebra. Denote by $\text{Inv}_0(A)$ the connected component of the set of invertible elements which contains the identity of A . Let $x, y \in A$ and let $\lambda \notin \text{sp}(x) \cup \text{sp}(y)$. Then $\lambda - x$ and $\lambda - y$ are invertible. Denote by $[\lambda - x]$ the corresponding element in $K_1(A)$. So $[\lambda - x] = [\lambda - y]$ means that they represent the same element in $K_1(A)$. In the case that A is of stable rank one, $[\lambda - x] = [\lambda - y]$ is equivalent to $(\lambda - x)^{-1}(\lambda - y) \in \text{Inv}_0(A)$.

Definition 2.5. Let A be a C^* -algebra and let $a, b \in A_+$ be two positive elements. We write $a \lesssim b$ if there is a sequence of elements $\{x_n\} \subset A$ such that

$$x_n^* b x_n \rightarrow a$$

as $n \rightarrow \infty$. If $a \lesssim b$ and $b \lesssim a$, then we write $[a] = [b]$ and say that a and b are equivalent in Cuntz semi-group.

If $p, q \in A$ are two projections, then $p \lesssim q$ means that there is a partial isometry $w \in A$ such that $w^* w = p$ and $w w^* \leq q$.

Definition 2.6. Let $\epsilon > 0$. Denote by f_ϵ the continuous function on $[0, \infty)$ such that $0 \leq f_\epsilon \leq 1$; $f_\epsilon(t) = 1$ if $t \in [\epsilon, \infty)$ and $f_\epsilon(t) = 0$ if $t \in [0, \epsilon/2]$ and $f_\epsilon(t)$ is linear in $(\epsilon/2, \epsilon)$.

Let $b \in A_+$ defined

$$d_\tau(b) = \lim_{\epsilon \rightarrow 0} \tau(f_\epsilon(b))$$

for $\tau \in T(A)$.

A is said to have strict comparison for positive elements, if

$$d_\tau(a) < d_\tau(b) \text{ for all } \tau \in T(A)$$

implies that $a \lesssim b$. In this paper, we mainly study those C^* -algebras A which have real rank zero, stable rank one and weak unperforated $K_0(A)$. Such C^* -algebras always have strict comparison for positive elements. When A has tracial rank zero (see 3.6.2 of [18]), we write $\text{TR}(A) = 0$. If A is a unital simple C^* -algebra with $\text{TR}(A) = 0$, then A has real rank zero, stable rank one and weakly unperforated $K_0(A)$.

Definition 2.7. Let Ω be a compact metric space and let $O \subset \Omega$ be an open subset. Throughout this paper, f_O denotes a positive function with $0 \leq f_O \leq 1$ whose support is exactly O , i.e., $f_O(t) > 0$ for all $t \in O$ and $f_O(t) = 0$ for all $t \notin O$. If $x, y \in A$ are two normal elements with $X = \text{sp}(x)$ and $Y = \text{sp}(y)$, we let $\Omega = X \cup Y$. Let $\varphi_X, \psi_Y : C(\Omega) \rightarrow A$ be unital homomorphisms defined by $\varphi_X(f) = f(x)$ and $\psi_Y(f) = f(y)$ for all $f \in C(\Omega)$.

Definition 2.8. Let $W(A)$ be the Cuntz semi-group which are equivalence classes of positive elements in $M_\infty(A)$.

Definition 2.9. Let A be a unital C^* -algebra and Ω be a compact metric space. Denote by $\text{Hom}_1(C(\Omega), A)$ the set of all unital homomorphisms κ from $C(\Omega)$ into A .

Definition 2.10. Let $O \subset \Omega$ be an open subset. Given $\kappa \in \text{Hom}_1(C(\Omega), A)$, $[\kappa(f_O)]$ does not depend on the choice of f_O . If $\kappa_1, \kappa_2 \in \text{Hom}_1(C(\Omega), A)$, define

$$D_c(\kappa_1, \kappa_2) = \sup\{\inf\{d > 0 : \kappa_1(f_O) \lesssim \kappa_2(f_{O_d})\} : O \subset \Omega, \text{ open}\}. \quad (\text{e } 2.4)$$

Remark 2.11. The definition of $D_c(\cdot, \cdot)$ is not symmetric a priori. However, when A is a unital simple C^* -algebra with stable rank one, then the definition in (e 2.4) is in fact symmetric.

Moreover, in general, $W(C(\Omega))$ is not determined by open subsets of Ω . In the definition it would required that $\kappa_1([f_O]) \lesssim \kappa_2([f_{O_d}])$, for all $f \in M_\infty(C(\Omega))_+$ whose supports in O and all $f_{O_d} \in M_\infty(C(\Omega))_+$ with supports in O_d . We will not study these in full generality.

Definition 2.12. Let A be a unital C^* -algebra and let Ω be a compact metric space. For $\kappa_1, \kappa_2 \in \text{Hom}_1(C(\Omega), A)$ we write $\kappa_1 \sim \kappa_2$ if

$$[\kappa_1(f_O)] = [\kappa_2(f_O)] \text{ in } W(A) \quad (\text{e } 2.5)$$

for all open subsets $O \subset \Omega$. It is easy to see that “ \sim ” is an equivalence relation. Put

$$H_{c,1}(C(\Omega), A) = \text{Hom}_1(C(\Omega), A)_+ / \sim.$$

It follows from [26] that if $\kappa_1 \sim \kappa_2$ then they induce the same semi-group homomorphisms from $W(C(\Omega))$ into $W(A)$ if the covering dimension of Ω is at most two and the second co Homology subsets $\check{H}^2(X) = \{0\}$ for each compact subsets X . This is particular true when Ω is a compact subset of the plane which is the primary concern of this research.

Let $\varphi \in \text{Hom}_1(C(\Omega), A)$. Then $\ker \varphi = \{f \in C(\Omega) : f|_X = 0\}$ for some compact subset $X \subset \Omega$. The compact subset X is called the spectrum of φ . Sometimes, we denote $\varphi \in \text{Hom}_1(C(\Omega), A)$ with spectrum X by φ_X .

Definition 2.13. Let A be a unital C^* -algebra and let X and Y be two compact subsets of a compact metric space Ω . Suppose that $\varphi : C(X) \rightarrow A$ and $\psi : C(Y) \rightarrow A$ are two unital monomorphisms. We define $\varphi_X : C(\Omega) \rightarrow A$ by $\varphi \circ \pi_X$ and $\psi_Y : C(\Omega) \rightarrow A$ by $\psi \circ \pi_Y$, where π_X and π_Y are the projections onto X and Y , respectively.

Let $a, d \in A_+$ with $a, d \leq 1$. In the following as well as in the proof of 2.15 below, we will write $a \ll d$. if there are $b, c \in A_+$ with $0 \leq b, c \leq 1$ such that

$$ab = a, \quad b \lesssim c \text{ and } dc = c. \quad (\text{e } 2.6)$$

The following follows immediately from [27] (see also [25], and 4.2 and 4.3 of [28]).

Lemma 2.14. Suppose that A is a unital C^* -algebra with stable rank one, $0 \leq a, d \leq 1$ are elements in A such that

$$a \ll d. \quad (\text{e } 2.7)$$

Then $1 - d \lesssim 1 - a$.

Proof. There are $b, c \in A$ such that $0 \leq b, c \leq 1$, $ab = a$, $b \lesssim c$ and $cd = c$. Since $ab = a$, for any $1/2 > \epsilon > 0$, $af_\epsilon(b) = a$. Since $b \lesssim c$, by Proposition 2.4 of [27], there is a unitary $u \in A$ such that

$$u^* f_\epsilon(b) u \in \text{Her}(c). \quad (\text{e } 2.8)$$

It follows that $u^* a u = d^{1/2} u^* a u d^{1/2} \leq d$. Therefore

$$1 - d \leq 1 - u^* a u = u^* (1 - a) u. \quad (\text{e } 2.9)$$

It follows that $1 - d \lesssim 1 - a$. \square

Proposition 2.15. *Let A be a unital simple C^* -algebra with stable rank one and let Ω be a compact metric space. Then $(H_{c,1}(C(\Omega), A), D_c)$ is a metric space. That is: for any $\varphi_X, \varphi_Y, \varphi_Z \in H_{c,1}(C(\Omega), A)$,*

$$D_c(\varphi_X, \varphi_Y) = 0 \iff \varphi_X \sim \varphi_Y, \quad (\text{e 2.10})$$

$$D_c(\varphi_X, \varphi_Y) = D_c(\varphi_Y, \varphi_X), \quad (\text{e 2.11})$$

$$D_c(\varphi_X, \varphi_Z) \leq D_c(\varphi_X, \varphi_Y) + D_c(\varphi_Y, \varphi_Z). \quad (\text{e 2.12})$$

Proof. Let $d = D_c(\varphi_X, \varphi_Y)$. We will show that $D_c(\varphi_Y, \varphi_X) = d$. Suppose $O \subset \Omega$ is an open subset. For any $d > \epsilon > 0$, let

$$F = \{t : \text{dist}(t, O) \geq d + \epsilon\} \text{ and } K = F_{d+\epsilon}.$$

Define $f, g \in C(\Omega)$ as the following:

$$f(t) = 0 \text{ if } t \notin F_{\epsilon/8}, \ 0 < f(t) < 1 \text{ if } t \in F_{\epsilon/8} \setminus F \text{ and } f(t) = 1 \text{ if } t \in F \text{ and} \quad (\text{e 2.13})$$

$$g(t) = 0 \text{ if } t \notin K, \ 0 < g(t) < 1 \text{ if } t \in K \setminus F_{d+\epsilon/2} \text{ and } g(t) = 1, \text{ if } t \in F_{d+\epsilon/2}. \quad (\text{e 2.14})$$

Since

$$F_{\epsilon/8} \subset \overline{F}_{\epsilon/8} \subset F_{\epsilon/4} \subset (F_{\epsilon/4})_{d+\epsilon/16} \subset F_{d+5\epsilon/16} \subset \overline{F_{d+5\epsilon/16}} \subset F_{d+\epsilon/2} \subset \overline{F_{d+\epsilon/2}} \subset K \quad (\text{e 2.15})$$

$$\text{and } \varphi_X(f_{F_{\epsilon/4}}) \lesssim \varphi_Y(f_{(F_{\epsilon/4})_{d+\epsilon/16}}), \quad (\text{e 2.16})$$

we have

$$\varphi_X(f) << \varphi_Y(g). \quad (\text{e 2.17})$$

By 2.14,

$$1 - \varphi_Y(g) \lesssim 1 - \varphi_X(f). \quad (\text{e 2.18})$$

Note that

$$O \subset \{t : \text{dist}(t, F) \geq d + \epsilon\} = \{t : \text{dist}(t, F) < d + \epsilon\}^c \subset K^c.$$

It follows that, for any $t \in O$, $g(t) = 0$, which implies

$$f_O \leq 1 - g. \quad (\text{e 2.19})$$

Hence

$$\varphi_Y(f_O) \lesssim 1 - \varphi_Y(g). \quad (\text{e 2.20})$$

On the other hand, if $f(t) \neq 1$ or $t \notin F$, then $\text{dist}(t, O) < d + \epsilon$, or $t \in O_{d+\epsilon}$. Therefore $\text{Supp}(1 - f) \subset O_{d+\epsilon}$, so we may assume that

$$1 - f \leq f_{O_{d+\epsilon}} \quad (\text{e 2.21})$$

by choosing a right representation of $f_{O_{d+\epsilon}}$. It follows that

$$1 - \varphi_X(f) \lesssim \varphi_X(f_{O_{d+\epsilon}}). \quad (\text{e 2.22})$$

Combining (e 2.20), (e 2.18) and (e 2.22), we obtain

$$\varphi_Y(f_O) \lesssim \varphi_X(f_{O_{d+\epsilon}}) \quad (\text{e 2.23})$$

for all $\epsilon > 0$ and for all open subsets $O \subset \Omega$.

This implies that

$$D_c(\varphi_Y, \varphi_X) \leq d. \quad (\text{e 2.24})$$

By symmetry, this proves that (e 2.11) holds.

Suppose that $D_c(\varphi_X, \varphi_Y) = 0$. For any non-empty open subset O , any $r > 0$, recall that

$$O_{-r} = \{t : \text{dist}(t, O^c) > r\}.$$

Then there is $\delta > 0$ such that for any $r \in (0, \delta)$, $O_{-r} \neq \emptyset$. It is easy to show, for any $S \subset \Omega$, $(S_{-r})_r \subset S$. For any $\epsilon \in (0, \delta)$, then

$$\varphi_X(f_{O_{-\epsilon}}) \lesssim \varphi_Y(f_{(O_{-\epsilon})_\epsilon}) \lesssim \varphi_Y(f_O).$$

This shows that, for any $\sigma \in (0, \delta)$,

$$f_\sigma(\varphi_X(f_O)) \lesssim \varphi_Y(f_O). \quad (\text{e 2.25})$$

It follows from 2.4 of [27] that

$$\varphi_X(f_O) \lesssim \varphi_Y(f_O).$$

Similarly, by (e 2.11), we have $D_c(\varphi_Y, \varphi_X) = 0$, so

$$\varphi_Y(f_O) \lesssim \varphi_X(f_O),$$

so $[\varphi_X(O)] = [\varphi_Y(O)]$ and (e 2.10) holds.

Finally, suppose that

$$D_c(\varphi_X, \varphi_Y) = d_1, D_c(\varphi_Y, \varphi_Z) = d_2, D_c(\varphi_X, \varphi_Z) = d_3.$$

Then for any open O and any $\epsilon > 0$,

$$\varphi_X(f_O) \lesssim \varphi_Y(f_{O_{d_1+\epsilon}}) \lesssim \varphi_Z(f_{O_{d_1+d_2+2\epsilon}}).$$

Therefore

$$d_3 \leq d_1 + d_2$$

and (e 2.12) holds. □

Proposition 2.16. *Let A be a simple unital C^* -algebra with stable rank one, and let Ω be a compact metric space. Then, for any finite subset of projections $\mathcal{P} \subset C(\Omega)$, there exists a $\delta > 0$ satisfying the following: if $\varphi, \psi : C(\Omega) \rightarrow A$ are two unital homomorphisms such that*

$$D_c(\varphi, \psi) < \delta, \quad (\text{e 2.26})$$

then

$$[\varphi(p)] = [\psi(p)] \text{ in } W(A) \text{ for all } p \in \mathcal{P}. \quad (\text{e 2.27})$$

Moreover,

$$[\varphi(p)] = [\psi(p)] \text{ in } K_0(A) \text{ for all } p \in \mathcal{P}. \quad (\text{e 2.28})$$

Proof. Without loss of generality, we may assume that \mathcal{P} consists of mutually orthogonal non-zero projections. There are mutually disjoint clopen subsets $\{E_i : i = 1, 2, \dots, m\}$ such that $\mathcal{P} = \{p_i = \chi_{E_i}, i = 1, 2, \dots, m\}$. Let

$$d = \min_{1 \leq i \leq m} \{\text{dist}(E_i, X \setminus E_i)\}.$$

Now choose $0 < \delta < d$. Note that, for any $d > r > 0$, $(E_i)_r = E_i$, $i = 1, 2, \dots, m$. If $D_c(\varphi, \psi) < \delta$, then for any i and $d > r > \delta$,

$$\varphi(f_{E_i}) \lesssim \psi(f_{(E_i)_r}) = \psi(f_{E_i})$$

which implies

$$[\varphi(p_i)] \lesssim [\psi(p_i)].$$

By symmetry, we have

$$[\psi(p_i)] \lesssim [\varphi(p_i)].$$

Thus we get $[\varphi(p_i)] = [\psi(p_i)]$, in $W(A)$ $i = 1, 2, \dots, m$. □

Lemma 2.17. *Let A be a unital simple C^* -algebra of (stable rank one) and let Ω be a compact metric space. For any $\epsilon > 0$, there exists $\delta > 0$ and a finite subset $\mathcal{F} \subset C(\Omega)$ satisfying the following:*

Suppose that $\varphi, \psi, \rho : C(\Omega) \rightarrow A$ are three unital homomorphisms such that

$$\|\varphi(f) - \psi(f)\| < \delta \text{ for all } f \in \mathcal{F}, \tag{e 2.29}$$

then

$$|D_c(\varphi, \rho) - D_c(\psi, \rho)| < \epsilon \tag{e 2.30}$$

Proof. Let $d = D_c(\varphi, \rho) \geq 0$. Let $\epsilon > 0$ be given.

Since Ω is compact, there are only finitely many open subsets $\{O_1, O_2, \dots, O_n\}$ such that, for any open subset $G \subset \Omega$, there is an integer i ,

$$G \subset O_i \subset G_\epsilon. \tag{e 2.31}$$

Let $g_i = f_{(O_i)_\epsilon}$, $i = 1, 2, \dots, n$.

It follows from [27] that there is $\delta > 0$ satisfying the following: If $h_1, h_2 : C(\Omega) \rightarrow A$ are two unital homomorphisms such that

$$\|h_1(g_i) - h_2(g_i)\| < \delta, \quad i = 1, 2, \dots, n, \tag{e 2.32}$$

then

$$f_\epsilon(h_1(g_i)) \lesssim h_2(g_i) \tag{e 2.33}$$

$i = 1, 2, \dots, n$.

Choose this δ . Let $\mathcal{F} = \{g_i : i = 1, 2, \dots, n\}$. Let $G \subset \Omega$ be an open subset. There is an integer i such that

$$G \subset O_i \subset G_\epsilon. \tag{e 2.34}$$

If

$$\|\varphi(g_i) - \psi(g_i)\| < \delta, \quad i = 1, 2, \dots, n, \tag{e 2.35}$$

then

$$f_\epsilon(\psi(g_i)) \lesssim \varphi(g_i). \quad (\text{e 2.36})$$

$$(\text{e 2.37})$$

Since the support of $f_\epsilon(g_i)$ contains O_i , $G \subset O_i$ and $O_i \subset G_\epsilon$, we obtain that

$$\psi(f_G) \lesssim \psi(f_\epsilon(g_i)) \lesssim \varphi(g_i) \lesssim \varphi(f_{G_{2\epsilon}}) \lesssim \rho(f_{G_{d+3\epsilon}}), \quad (\text{e 2.38})$$

Since this holds for all open sets $G \subset \Omega$, we conclude that

$$D_c(\psi, \rho) \leq d + 3\epsilon = D_c(\varphi, \rho) + 3\epsilon. \quad (\text{e 2.39})$$

By symmetry,

$$D_c(\varphi, \rho) \leq D_c(\psi, \rho) + 3\epsilon. \quad (\text{e 2.40})$$

Lemma follows. \square

Definition 2.18. Let A be a unital C^* -algebra and let Ω be a compact metric space. Fix $\kappa \in H_{c,1}(C(\Omega), A)$ and $\epsilon > 0$. Let X be the spectrum of κ . We say κ has a *finite ϵ -approximation* in $H_{c,1}(C(\Omega), A)$, if there is a finite subset $F \subset X$ and $\varphi_F \in \text{Hom}_1(C(\Omega), A)_+$ with the spectrum F such that

$$D_c(\kappa, \varphi_F) < \epsilon. \quad (\text{e 2.41})$$

Note that $[\varphi_F(f_O)]$ can be represented by a projection $p \in A$.

Let A be a unital separable simple C^* -algebra with real rank zero, stable rank one and with weakly unperforated $K_0(A)$. We show that (see 4.12 below), if $\kappa \in H_{c,1}(C(\Omega), A)$ is induced by a homomorphism $h : C(\Omega) \rightarrow A$, then κ has a finite ϵ -approximation for any $\epsilon > 0$. Warning: a homomorphism φ has a finite ϵ -approximation in $H_{c,1}(C(\Omega), A)$ does not imply that it is close to a homomorphisms with finite spectrum.

Definition 2.19. Let A be a unital C^* -algebra with $T(A) \neq \emptyset$. Let Ω be a compact metric space, and $X, Y \subset \Omega$ be two compact subsets. Let $\varphi_X : C(X) \rightarrow A$ and $\psi_Y : C(Y) \rightarrow A$ be two unital monomorphisms. Let $\pi_X : C(\Omega) \rightarrow C(X)$ and $\pi_Y : C(\Omega) \rightarrow C(Y)$ be the quotient maps. Define $\varphi_X = \varphi \circ \pi_X$ and $\psi_Y = \psi \circ \pi_Y$. For each open subset $O \subset \Omega$, define

$$r_O(\varphi_X, \psi_Y) = \inf\{r > 0 : d_\tau(\varphi_X(f_O)) \leq d_\tau(\psi_Y(f_{O_r})) \text{ for all } \tau \in T(A)\} \text{ and} \quad (\text{e 2.42})$$

$$r_O^+(\varphi_X, \psi_Y) = \inf\{r > 0 : d_\tau(\varphi_X(f_O)) < d_\tau(\psi_Y(f_{O_r})) \text{ for all } \tau \in T(A)\}. \quad (\text{e 2.43})$$

Define

$$D_T(\varphi_X, \psi_Y) = \sup\{r_O(\varphi_X, \psi_Y) : O \text{ open}\}. \quad (\text{e 2.44})$$

Put

$$a(\varphi_X, \psi_Y) = \sup\{\text{dist}(\zeta, X) : \zeta \in Y\} \text{ and } b(\varphi_X, \psi_Y) = \sup\{\text{dist}(\xi, Y) : \xi \in X\}.$$

Define

$$D^T(\varphi_X, \psi_Y) = \max\{a(\varphi_X, \psi_Y), \sup\{r_O^+(\varphi_X, \psi_Y) : O \text{ open and } O \cap X \neq X\}\}. \quad (\text{e 2.45})$$

Note that, if $X \subset O$, then $d_\tau(\varphi_X(f_O)) = 1$ for all $\tau \in T(A)$. Therefore

$$D_T(\varphi_X, \psi_Y) \geq a(\varphi_X, \psi_Y). \quad (\text{e 2.46})$$

Since X is compact, there is $\xi \in X$ such that $b(\varphi_X, \psi_Y) = \text{dist}(\xi, Y)$. If $\epsilon > 0$ and $O(\xi, \epsilon)$ is the open ball with center at ξ and radius ϵ , then

$$r_{O(\xi, \epsilon)}(\varphi_X, \psi_Y) \geq b(\varphi_X, \psi_Y) - \epsilon. \quad (\text{e 2.47})$$

It follows that $D_T(\varphi_X, \psi_Y) \geq \max\{a(\varphi_X, \psi_Y), b(\varphi_X, \psi_Y)\} = d_H(X, Y)$, where $d_H(X, Y)$ is the Hausdorff distance between X and Y .

Lemma 2.20. *Let A be a unital simple C^* -algebra with $T(A) \neq \emptyset$ and let $O \subset \Omega$ be an open set with $O \cap X \neq X$. If $r > D^T(\varphi_X, \psi_Y)$, then*

$$\inf\{d_\tau(\psi_Y(f_{O_r})) - d_\tau(\varphi_X(f_O)) : \tau \in T(A)\} > 0. \quad (\text{e 2.48})$$

Proof. Put $d = D^T(\varphi_X, \psi_Y)$ and $\eta = (1/4)(r - d) > 0$. Let $f_1 \in C(\Omega)$ with $0 \leq f_1 \leq 1$ such that $f_1(t) = 1$ if $t \in O$ and $f_1(t) = 0$ if $t \notin O_\eta$. Let $f_2 \in C(\Omega)$ with $0 \leq f_2 \leq 1$ such that $f_2(t) = 1$ if $t \in O_{d+2\eta}$ and $f_2(t) = 0$ if $t \notin O_r$. Then

$$d_\tau(\psi_Y(f_{O_r})) \geq \tau(\psi_Y(f_2)) \geq d_\tau(\psi_Y(f_{O_{d+2\eta}})) \quad (\text{e 2.49})$$

$$> d_\tau(\varphi_X(f_{O_\eta})) \geq \tau(\varphi_X(f_1)) \geq d_\tau(\varphi_X(f_O)) \quad (\text{e 2.50})$$

for all $\tau \in T(A)$. Since $T(A)$ is compact, we conclude that

$$\inf\{\tau(\psi_Y(f_2)) - \tau(\varphi_X(f_1)) : \tau \in T(A)\} > 0. \quad (\text{e 2.51})$$

Therefore

$$\inf\{d_\tau(\psi_Y(f_{O_r})) - d_\tau(\varphi_X(f_O)) : \tau \in T(A)\} \quad (\text{e 2.52})$$

$$\geq \inf\{\tau(\psi_Y(f_2)) - \tau(\varphi_X(f_1)) : \tau \in T(A)\} > 0. \quad (\text{e 2.53})$$

□

Proposition 2.21. *Let A be a unital C^* -algebra with $T(A) \neq \emptyset$. If A has the strict comparison for positive elements. Let $\varphi_X, \varphi_Y, \varphi_Z$ be three unital homomorphisms from $C(\Omega)$ into A . Then*

$$D_T(\varphi_X, \varphi_Y) \leq D_c(\varphi_X, \varphi_Y) \leq D^T(\varphi_X, \varphi_Y), \quad (\text{e 2.54})$$

$$D_T(\varphi_X, \varphi_Y) = D_T(\varphi_Y, \varphi_X), \quad (\text{e 2.55})$$

$$D^T(\varphi_X, \varphi_Y) = D^T(\varphi_Y, \varphi_X), \quad (\text{e 2.56})$$

$$D_T(\varphi_X, \varphi_Z) \leq D_T(\varphi_X, \varphi_Y) + D_T(\varphi_Y, \varphi_Z), \quad (\text{e 2.57})$$

$$D^T(\varphi_X, \varphi_Z) \leq D^T(\varphi_X, \varphi_Y) + D^T(\varphi_Y, \varphi_Z). \quad (\text{e 2.58})$$

If X or Y is connected, then

$$D_T(\varphi_X, \varphi_Y) = D_c(\varphi_X, \varphi_Y) = D^T(\varphi_X, \varphi_Y). \quad (\text{e 2.59})$$

Proof. If $D_c(\varphi_X, \varphi_Y) < d$, then for any open O ,

$$\varphi_X(f_O) \lesssim \varphi_Y(f_{O_d}).$$

So for any $\tau \in T(A)$,

$$d_\tau(\varphi_X(f_O)) \leq d_\tau(\varphi_Y(f_{O_d})).$$

This implies the first inequality of (e 2.54).

If $D^T(\varphi_X, \varphi_Y) < d$, then for any open subset O with $O \cap X \neq X$, by 2.20,

$$\inf\{d_\tau(\varphi_Y(f_{O_d})) - d_\tau(\varphi_X(f_O)) : \tau \in T(A)\} > 0. \quad (\text{e 2.60})$$

Since A has strict comparison for positive elements, we have

$$\varphi_X(f_O) \lesssim \varphi_Y(f_{O_d}).$$

If $O \cap X = X$, then $X \subset O$. Since $d > a(\varphi_X, \psi_Y)$, we have that $Y \subset X_d \subset O_d$. It follows $\varphi_X(f_O) \lesssim \varphi_Y(f_{O_d})$. Thus we get the second inequality of (e 2.54).

To show (e 2.55), let $D_T(\varphi_X, \varphi_Y) \leq d$. It suffices to show that $D_T(\varphi_Y, \varphi_X) \leq d$. Suppose $O \subset \Omega$ is an open subset. For any $\epsilon > 0$, let

$$F = \{t : \text{dist}(t, O) \geq d + \epsilon\} \text{ and } K = F_{d+\epsilon}.$$

Define $f, g \in C(\Omega)$ with $f(t) = 0$ if $t \notin F_{\epsilon/2}$, $0 < f(t) < 1$, if $t \in F_{\epsilon/2} \setminus F_{\epsilon/4}$, $f(t) = 1$ if $t \in F_{\epsilon/4}$ and $g(t) = 0$ if $t \notin K$, $0 < g(t) < 1$, if $t \in K \setminus F_{d+\epsilon/2}$, $g(t) = 1$ if $t \in F_{d+\epsilon/2}$. So, if $1 - f(t) \neq 0$, then $t \notin F$. Hence $t \in O_{d+\epsilon}$. Therefore, by choosing a right $f_{O_{d+\epsilon}}$, we may assume $1 - f \leq f_{O_{d+\epsilon}}$.

Then

$$(F_{\epsilon/2})_{d+\epsilon/2} \subset F_{d+\epsilon} = K. \quad (\text{e 2.61})$$

By the definition,

$$d_\tau(\varphi_X(f)) = d_\tau(\varphi_X(f_{F_{\epsilon/2}})) \leq d_\tau(\varphi_Y(f_{(F_{\epsilon/2})_{d+\epsilon/2}})) \leq d_\tau(\varphi_Y(f_K)) = d_\tau(\varphi_Y(g)). \quad (\text{e 2.62})$$

Thus

$$1 - d_\tau(\varphi_Y(g)) \leq 1 - d_\tau(\varphi_X(f)). \quad (\text{e 2.63})$$

Since $f_O \leq 1 - g$, we have

$$\varphi_Y(f_O) \lesssim 1 - \varphi_Y(g). \quad (\text{e 2.64})$$

Hence

$$1 - d_\tau(\varphi_X(f)) \leq d_\tau(\varphi_X(f_{O_{d+\epsilon}})). \quad (\text{e 2.65})$$

Thus we have

$$d_\tau(\varphi_Y(f_O)) \leq d_\tau(\varphi_X(f_{O_{d+\epsilon}})) \quad (\text{e 2.66})$$

for all $\epsilon > 0$ and for all open subsets $O \subset \Omega$. Thus we get $D_T(y, x) \leq d$.

To show (e 2.56), let $d > D^T(\varphi_X, \varphi_Y)$ and let $O \subset \Omega$. If $O \cap Y \neq Y$, we have

$$d_\tau(\varphi_X(f_{F_{\epsilon/2}})) < d_\tau(\varphi_Y(f_{(F_{\epsilon/2})_{d+\epsilon/2}})). \quad (\text{e 2.67})$$

We will follow the proof of (e 2.55). By (e 2.67), instead of “ \leq ” we will have “ $<$ ” in (e 2.62). It follows, as in the proof of (e 2.55),

$$\sup\{r_O^+(\varphi_Y, \varphi_X) : O \text{ open and } O \cap Y \neq Y\} \leq d.$$

Since $a(\varphi_Y, \varphi_X) \leq D_T(\varphi_Y, \varphi_X) = D_T(\varphi_X, \varphi_Y) < d$, then

$$D^T(\varphi_Y, \varphi_X) = \max\{a(\varphi_Y, \varphi_X), \sup\{r_O^+(\varphi_Y, \varphi_X) : O \text{ open and } O \cap Y \neq Y\}\} \leq d. \quad (\text{e 2.68})$$

We get (e 2.56).

Now we turn to (e 2.57). If $c_1 = D_T(\varphi_X, \varphi_Y)$, $c_2 = D_T(\varphi_Y, \varphi_Z)$, $c_3 = D^T(\varphi_X, \varphi_Y)$ and $c_4 = D^T(\varphi_X, \varphi_Y)$, then for any $\epsilon > 0$, any open set O ,

$$d_\tau(\varphi_X(f_O)) \leq d_\tau(\varphi_Y(f_{O_{c_1+\epsilon}})) \leq d_\tau(\varphi_Z(f_{O_{c_1+c_2+2\epsilon}}))$$

and

$$d_\tau(\varphi_X(f_O)) < d_\tau(\varphi_Y(f_{O_{c_3+\epsilon}})) < d_\tau(\varphi_Z(f_{O_{c_3+c_4+2\epsilon}}))$$

If

$$a(\varphi_X, \varphi_Y) = d_1, a(\varphi_Y, \varphi_Z) = d_2,$$

then $Z \subset Y_{d_2} \subset (X_{d_1})_{d_2} \subset X_{d_1+d_2}$, so

$$a(\varphi_X, \varphi_Z) = \inf\{r > 0 : Z \subset X_r\} \leq d_1 + d_2 = a(\varphi_X, \varphi_Y) + a(\varphi_Y, \varphi_Z).$$

From these, we obtain (e 2.57) and (e 2.58).

To show (e 2.59), assume X is connected and $D_T(\varphi_X, \varphi_Y) = d$. It suffices to show $D^T(\varphi_X, \varphi_Y) \leq d$. For any open O with $O \cap X \neq X$, since X is connected, there is $\delta > 0$ such that for any $0 < \epsilon < \delta$, $O \cap X \neq O_{\epsilon/2} \cap X$. So, since A is simple, for any $\tau \in T(A)$,

$$\tau(\varphi_X(f_O)) < \tau(\varphi_X(f_{O_{\epsilon/2}})) \leq \tau(\varphi_Y(f_{O_{d+\epsilon}})). \quad (\text{e 2.69})$$

On the other hand, since $a(\varphi_X, \varphi_Y) \leq D_T(\varphi_X, \varphi_Y)$, by definition, $D^T(\varphi_X, \varphi_Y) \leq d$. This ends the proof of (e 2.59). \square

Note there exists C^* -algebra A , $D_T(\varphi, \psi) = D_c(\varphi, \psi)$, even when neither X nor Y are connected.

Proposition 2.22. *Let A be a unital simple C^* -algebra with stable rank one, $\ker \rho_A(K_0(A)) = \{0\}$ and with strict comparison for positive elements. Suppose that A has a unique tracial state. Then*

$$D_T(\varphi, \psi) = D_c(\varphi, \psi).$$

Proof. Let $\varphi, \psi : C(\Omega) \rightarrow A$ be two unital homomorphisms with spectrum X and Y , respectively, and let τ be the unique tracial state of A . By (e 2.54) of 2.21, it suffices to show that $D_T(\varphi, \psi) \geq D_c(\varphi, \psi)$. Let $d = D_T(\varphi, \psi)$ and $d_1 > d$. For any open subset $O \subset \Omega$,

$$d_\tau(\varphi(f_O)) \leq d_\tau(\psi(f_{O_{d_1}})). \quad (\text{e 2.70})$$

If inequality holds in (e 2.70), then by the strict comparison,

$$\varphi(f_O) \lesssim \psi(f_{O_{d_1}}). \quad (\text{e 2.71})$$

Otherwise, suppose that equality holds in (e 2.70).

If, for every $1/2 > \epsilon > 0$,

$$d_\tau(f_\epsilon(\varphi(f_O))) = d_\tau(\varphi(f_\epsilon(f_O))) < d_\tau(\varphi(f_O)) = d_\tau(\psi(f_{O_{d_1}})), \quad (\text{e 2.72})$$

by the strict comparison again,

$$\varphi(f_\epsilon(f_O)) \lesssim \psi(f_{O_{d_1}}) \text{ for all } 1/2 > \epsilon > 0. \quad (\text{e 2.73})$$

It follows that $\varphi(f_O) \lesssim \psi(f_{O_{d_1}})$.

Otherwise there is an $1/2 > \epsilon > 0$ such that

$$d_\tau(f_\epsilon(\varphi(f_O))) = d_\tau(\varphi(f_O)). \quad (\text{e 2.74})$$

Since A is simple, we conclude that, for $\delta > 0$,

$$O \cap X = O_{-\delta} \cap X = \{\xi \in X : \text{dist}(x, O^c) > \delta\}, \quad (\text{e 2.75})$$

which implies that $O \cap X$ is a clopen set relative to X . Let $q = \varphi(f_O)$ which is then a projection in this case. We also have, for any $d < d_2 < d_1$,

$$d_\tau(\varphi(f_O)) = d_\tau(\varphi_Z(f_{O_{d_2}})) = d_\tau(\varphi_Y(f_{O_{d_1}})). \quad (\text{e 2.76})$$

The same argument above shows that $O_{d_2} \cap Y = O_{d_1} \cap Y$. It follows that $p = \psi(f_{O_{d_1}})$ is a projection. Since $\ker \rho_A(K_0(A)) = \{0\}$, and $\tau(p) = \tau(q)$,

$$\varphi(f_O) = q \sim p = \psi(f_{O_{d_1}}). \quad (\text{e 2.77})$$

It follows that $D_c(\varphi, \psi) \leq d_1$ for all $d_1 > d$. This completes the proof. \square

Definition 2.23. For $\varphi, \psi \in \text{Hom}(C(\Omega), A)$, let

$$\delta_T(\varphi, \psi) = \inf\{r > 0 : d_\tau(\varphi(f_O)) \leq d_\tau(\psi(f_{O_r})), \text{ for all } O = O(\lambda, d), \tau \in T(A)\},$$

$$\delta_c(x, y) = \inf\{r > 0 : \varphi(f_O) \lesssim \psi(f_{O_r}), \text{ for all } O = O(\lambda, d), \}$$

and

$$d_T(\varphi, \psi) = \max\{\delta_T(\varphi, \psi), \delta_T(\psi, \varphi)\},$$

$$d_c(\varphi, \psi) = \max\{\delta_c(\varphi, \psi), \delta_c(\psi, \varphi)\}.$$

For any normal elements $x, y \in A$, if $X = \text{sp}(x)$, $Y = \text{sp}(y)$, $\Omega = X \cup Y$ and φ_X, φ_Y are defined by $\varphi_X(f) = f(x)$, $\varphi_Y(f) = f(y)$ for any $f \in C(\Omega)$, define

$$d_T(x, y) = d_T(\varphi_X, \varphi_Y), \quad d_c(x, y) = d_c(\varphi_X, \varphi_Y).$$

Proposition 2.24. Let Ω be a compact metric space and A be a unital simple C^* -algebra. Suppose that $\varphi_X, \varphi_Y, \varphi_Z : C(\Omega) \rightarrow A$ are unital homomorphisms with spectrum $X, Y, Z \subset \Omega$, respectively. Then

$$d_T(\varphi_X, \varphi_Y) = d_T(\varphi_Y, \varphi_X), \quad d_c(\varphi_X, \varphi_Y) = d_c(\varphi_Y, \varphi_X) \quad (\text{e 2.78})$$

$$d_T(\varphi_X, \varphi_Y) \geq d_H(X, Y), \quad (\text{e 2.79})$$

$$d_T(\varphi_X, \varphi_Y) \leq d_c(\varphi_X, \varphi_Y) \leq D_c(\varphi_X, \varphi_Y), \quad (\text{e 2.80})$$

$$d_T(\varphi_X, \varphi_Y) \leq d_T(\varphi_X, \varphi_Z) + d_T(\varphi_Z, \varphi_Y), \quad (\text{e 2.81})$$

$$d_c(\varphi_X, \varphi_Y) \leq d_c(\varphi_X, \varphi_Z) + d_c(\varphi_Z, \varphi_Y). \quad (\text{e 2.82})$$

Proof. The identities in (e 2.78) follows from the definition. The inequality in (e 2.79) also follows from the definition immediately since A is assumed to be simple. If $d_c(\varphi_X, \psi_Y) = r$, then for any $\xi \in \Omega$, any $d > 0$, any $\epsilon > 0$,

$$\varphi_X(f_{O(\xi, d)}) \lesssim \varphi_Y(f_{O(\xi, d+r+\epsilon)}).$$

It follows that, for any $\tau \in T(A)$,

$$d_\tau(\varphi_X(f_{O(\xi, d)})) \leq d_\tau(\varphi_Y(f_{O(\xi, d+r+\epsilon)})).$$

This implies $d_T(\varphi_X, \psi_Y) \leq d_c(\varphi_X, \psi_Y)$. It is obvious that $d_c(\varphi_X, \psi_Y) \leq D_c(\varphi_X, \psi_Y)$. So (e 2.80) holds. The proofs of (e 2.81) and (e 2.82) are similar, we show (e 2.81) only.

If $d_T(\varphi_X, \varphi_Z) < d_1, d_T(\varphi_Z, \varphi_Y) < d_2$, then for any $\xi \in \mathbb{C}$, any $d > 0$, any $\epsilon > 0$,

$$d_T(\varphi_X(f_{O(\xi, d)})) \leq d_T(\rho_Z(f_{O(\xi, d+d_1+\epsilon)})) \leq d_T(\psi_Y(f_{O(\xi, d+d_1+d_2+2\epsilon)})),$$

therefore $d_T(\varphi_X, \psi_Y) \leq d_1 + d_2$, this implies (e 2.81) holds. \square

Remark 2.25. There are examples such that $d_T(x, y) = D_c(x, y)$.

Let $n \geq 4$ be an integer. Let $X = \{e^{k\pi i/n} : 0 \leq k \leq n\}$ and $Y = rX = \{re^{k\pi i/n} : 0 \leq k \leq n\}$ for some $0 < r < 1$. Let A be any unital simple C^* -algebra with $T(A) \neq \emptyset$ which has n mutually orthogonal non-zero projections $\{e_1, e_2, \dots, e_n\}$ such that $\sum_{k=1}^n e_i = 1$. Define $x = \sum_{k=1}^n e^{(k-1)\pi i/n} e_i$ and $y = \sum_{k=1}^n re^{(k-1)\pi i/n} e_i$. Then one computes that

$$D_T(x, y) = 1 - r = d_T(x, y) = d_c(x, y) = D_c(x, y). \quad (\text{e 2.83})$$

On the other hand, of course, there are also examples that $d_T(x, y) < D_T(x, y)$. Let $\{e_1, e_2, e_3\}$ be mutually orthogonal and equivalent projections with $e_1 + e_2 + e_3 = 1$,

$$x = -e_1 + e_3, y = -ie_1 + ie_3.$$

Then

$$d_T(x, y) = d_c(x, y) = 1 < \sqrt{2} = D_T(x, y) = D_c(x, y).$$

In the case that A has stable rank one, strict comparison and $\ker \rho_A = \{0\}$, then $D_T(\cdot, \cdot)$ is a distance on $H_{c,1}(C(\Omega), A)$. In general, because of the definition of $H_{c,1}(C(\Omega), A)$, D^T, d_T, d_c are not a distance on $H_{c,1}(C(\Omega), A)$.

3 Distance between unitary orbits of normal elements with trivial K_1

Let \mathbb{Z}^k be the direct sum of n copies of the abelian group \mathbb{Z} . Put

$$\mathbb{Z}_+^k = \{(n_1, n_2, \dots, n_k) : n_j \geq 0 : j = 1, 2, \dots, k\}. \quad (\text{e 3.84})$$

It is well known that $(\mathbb{Z}^k, \mathbb{Z}_+^k)$ is an unperforated (partially) ordered group with the Riesz interpolation property. Let $R \subset \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ be a subset and let $A \subset \{1, 2, \dots, m\}$. Define $R_A \subset \{1, 2, \dots, n\}$ to be the subset of those j 's such that $(i, j) \in R$, for some $i \in A$.

Lemma 3.1. *If $\{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n \subset \mathbb{Z}_+^k$ with $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$, and $R \subset \{1, \dots, m\} \times \{1, \dots, n\}$ satisfying: for any $A \subset \{1, \dots, m\}$,*

$$\sum_{i \in A} a_i \leq \sum_{j \in R_A} b_j, \quad (\text{e 3.85})$$

then there are $\{c_{ij}\} \subset \mathbb{Z}_+^k$ such that

$$\sum_{j=1}^n c_{ij} = a_i, \sum_{i=1}^m c_{ij} = b_j, \text{ for all } i, j \quad (\text{e 3.86})$$

and

$$c_{ij} = 0 \text{ unless } (i, j) \in R. \quad (\text{e 3.87})$$

Proof. Write

$$a_i = (a_i(1), a_i(2), \dots, a_i(k)) \text{ and } b_j = (b_j(1), b_j(2), \dots, b_j(k)), \quad (\text{e 3.88})$$

$i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$.

It follows from Lemma 1.2 of [12] that, for each s ($s = 1, 2, \dots, k$), there are $c_{i,j}(s) \in \mathbb{Z}_+$ such that

$$\sum_{j=1}^n c_{ij}(s) = a_i(s), \sum_{i=1}^m c_{ij}(s) = b_j(s), \text{ for all } i, j. \quad (\text{e 3.89})$$

and

$$c_{ij}(s) = 0 \text{ unless } (i, j) \in R. \quad (\text{e 3.90})$$

Define

$$c_{ij} = (c_{ij}(1), c_{ij}(2), \dots, c_{ij}(k)), \quad i = 1, 2, \dots, m \text{ and } j = 1, 2, \dots, n \quad (\text{e 3.91})$$

Note that

$$c_{ij} = 0 \text{ unless } (i, j) \in R. \quad (\text{e 3.92})$$

We also have

$$\sum_{i=1}^m c_{ij} = a_i \text{ and } \sum_{j=1}^n c_{ij} = b_j. \quad (\text{e 3.93})$$

□

Lemma 3.2. *Let (G, G_+) be a countable torsion free unperforated partially ordered abelian group with the Riesz interpolation property. If $\{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n \subset G_+$ with $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$, and $R \subset \{1, \dots, m\} \times \{1, \dots, n\}$ satisfying: for any $A \subset \{1, \dots, m\}$,*

$$\sum_{i \in A} a_i \leq \sum_{j \in R_A} b_j, \quad (\text{e 3.94})$$

then there are $\{c_{ij}\} \subset G_+$ such that

$$\sum_{j=1}^n c_{ij} = a_i, \sum_{i=1}^m c_{ij} = b_j, \text{ for all } i, j \quad (\text{e 3.95})$$

and

$$c_{ij} = 0 \text{ unless } (i, j) \in R. \quad (\text{e 3.96})$$

Proof. Let G be a countable unperforated ordered Riesz group. It follows from [8] that $G = \lim_{n \rightarrow \infty} (G_n, h_n)$ with G_n is order isomorphic to $\mathbb{Z}_+^{r(n)}$ (with positive cone $\mathbb{Z}_+^{r(n)}$), where $h_n : G_n \rightarrow G_{n+1}$. Denote by $h_{n,n+k} = h_{n+k} \circ h_{n+(k-1)} \circ \dots \circ h_n : G_n \rightarrow G_{n+k}$, $k = 1, 2, \dots, n = 1, 2, \dots$, and denote by $h_{n,\infty} : G_n \rightarrow G$ the homomorphism induced by the inductive limit system. Moreover, $G_+ = \cup_{n=1}^{\infty} h_{n,\infty}((G_n)_+)$. For each $A \subset \{1, 2, \dots, m\}$, denote by

$$g_A = \sum_{j \in R_A} b_j - \sum_{i \in A} a_i \quad (\text{e 3.97})$$

Note that there are no more than 2^m many A 's. There exists $n_1 > 0$ such that there are

$$a_{i,k}, b_{j,k} \in (G_k)_+ \text{ and} \quad (\text{e 3.98})$$

$$g_{A,k} \in (G_k)_+ \text{ for all } i, j \text{ and } A \subset \{1, 2, \dots, m\} \quad (\text{e 3.99})$$

such that, for $k_1 > k \geq n_1$, $h_{k,k_1}(a_{i,k}) = a_{i,k_1}$, $h_{k,k_1}(b_{j,k}) = b_{j,k_1}$, $h_{k,k_1}(g_{A,k}) = g_{A,k_1}$, $h_{k,\infty}(a_{i,k}) = a_i$, $h_{k,\infty}(b_{j,k}) = b_j$ and $h_{k,\infty}(g'_A) = g_A$. Note, since each G_k is isomorphic to $\mathbb{Z}^{r(n)}$, there is an integer $n_2 > n_1$ such that

$$\sum_{i=1}^m h_{n_1,n_2}(a_{i,n_1}) - \sum_{j=1}^n h_{n_1,n_2}(b_{j,n_1}) = 0 \text{ and} \quad (\text{e 3.100})$$

$$g_{A,n_2} = \sum_{j \in R_A} h_{n_1,n_2}(b_{j,n_1}) - \sum_{i \in A} h_{n_1,n_2}(a_{i,n_1}). \quad (\text{e 3.101})$$

Thus, we obtain, by applying 3.1, $c_{i,j,n_2} \in (G_{n_2})_+$, $(i, j) \in R$ such that

$$\sum_{j=1}^n c_{i,j,n_2} = a_{i,n_2} \text{ and } \sum_{i=1}^m c_{i,j,n_2} = b_{j,n_2}. \quad (\text{e 3.102})$$

Moreover,

$$c_{i,j,n_2} = 0 \text{ unless } (i, j) \in R. \quad (\text{e 3.103})$$

Define $c_{i,j} = h_{n_2,\infty}(c_{i,j,n_2})$. Then, $c_{i,j} \geq 0$ and by (e 3.102) and (e 3.103),

$$\sum_{j=1}^n c_{i,j} = a_i, \quad \sum_{i=1}^m c_{i,j} = b_j \text{ and} \quad (\text{e 3.104})$$

$$c_{i,j} = 0 \text{ unless } (i, j) \in R. \quad (\text{e 3.105})$$

□

Lemma 3.3. *Let (G, G_+) be a countable weakly unperforated partially ordered abelian group with the Riesz interpolation property. If $\{a_i\}_{i=1}^m, \{b_j\}_{j=1}^n \subset G_+$ with $\sum_{i=1}^m a_i = \sum_{j=1}^n b_j$, $R \subset \{1, \dots, m\} \times \{1, \dots, n\}$ satisfying: for any $A \subset \{1, \dots, m\}$,*

$$\sum_{i \in A} a_i \leq \sum_{j \in R_A} b_j, \quad (\text{e 3.106})$$

then there are $\{c_{ij}\} \subset G_+$ such that

$$\sum_{j=1}^n c_{ij} = a_i, \quad \sum_{i=1}^m c_{ij} = b_j, \text{ for all } i, j \quad (\text{e 3.107})$$

and

$$c_{ij} = 0 \text{ unless } (i, j) \in R. \quad (\text{e 3.108})$$

Proof. It follows from [9] that one may write

$$0 \rightarrow T \rightarrow G \rightarrow G_0 \rightarrow 0,$$

where G_0 is a countable unperforated ordered group with the Riesz interpolation property and T is a countable abelian torsion group. Moreover, $g \in G_+ \setminus \{0\}$ if and only if $d(g) \in (G_0)_+$,

where $d : G \rightarrow G_0$ is the quotient map. Furthermore, there exists a sequence of abelian groups S_n and T_n such that S_n is order isomorphic to $\mathbb{Z}^{r(n)}$ and $T_n = \mathbb{Z}/k(1, n)\mathbb{Z} \oplus \mathbb{Z}/k(2, n)\mathbb{Z} \oplus \cdots \oplus \mathbb{Z}/k(t(n), n)\mathbb{Z}$ such that $G = \lim_{n \rightarrow \infty} (S_n \oplus T_n, \iota_n)$, where $\iota_n : S_n \oplus T_n \rightarrow S_{n+1} \oplus T_{n+1}$. Denote by $\iota_{n, \infty} : S_n \oplus T_n \rightarrow G$. Note

$$(S_n \oplus T_n)_+ = \{(s, f) : s \in (S_n)_+ \setminus \{0\}\} \cup \{(0, 0)\}, \quad n = 1, 2, \dots,$$

and $G_+ = \lim_{n \rightarrow \infty} (S_n \oplus T_n)_+$. Let $\pi'_n : S_n \oplus T_n \rightarrow S_n$ and let $\pi''_n : S_n \oplus T_n \rightarrow T_n$ be the projection maps. Let $\iota'_n : S_n \rightarrow S_{n+1}$ be defined by $\iota'_n = \pi'_n \circ \iota_n|_{S_n}$. Let $F_n = \mathbb{Z}^{t(n)}$ and $\pi_n : F_n \rightarrow T_n$ be the quotient map. Define $H_n = S_n \oplus F_n$. Since F_n is free, there is a homomorphism $j_n : F_n \rightarrow F_{n+1}$ such that

$$\begin{array}{ccc} F_n & \xrightarrow{j_n} & F_{n+1} \\ \downarrow \pi_n & & \downarrow \pi_{n+1} \\ T_n & \xrightarrow{\iota_n} & T_{n+1} \end{array}$$

commutes. Since S_n is free, there is $h'_n : S_n \rightarrow F_{n+1}$ such that

$$\begin{array}{ccc} S_n & \xrightarrow{\iota_n} & S_{n+1} \oplus T_{n+1} \\ \downarrow h'_n & & \downarrow \pi''_{n+1} \\ F_{n+1} & \xrightarrow{\pi_{n+1}} & T_{n+1} \end{array}$$

commutes. Define $h_n : H_n \rightarrow H_{n+1}$ by

$$h_n|_{S_n} = \iota'_n \oplus h'_n \quad \text{and} \quad h_n|_{F_n} = j_n, \quad n = 1, 2, \dots$$

Define $(H_n)_+ = \{(s, f) : s \in (S_n)_+ \setminus \{0\}\} \cup \{(0, 0)\}$. Let $H = \lim_{n \rightarrow \infty} (H_n, h_n)$ and let $F = \lim_{n \rightarrow \infty} (F_n, h_n|_{F_n})$. Define $H_+ = \cup_{n=1}^{\infty} h_{n, \infty}(H_n)_+$, where $h_{n, \infty} : H_n \rightarrow H$ is the homomorphism induced by the inductive limit system. Define $d_1 : H \rightarrow H/F$. Then it is clear that H/F is order isomorphic to G_0 . Moreover, if $h \in H$, then $h \in H_+$ if and only if $d_1(h) \in (G_0)_+$. Therefore H is also a torsion free weakly unperforated ordered group with Riesz interpolation property.

Define $q_n : H_n \rightarrow S_n \oplus T_n$ by $q_n|_{S_n} = \text{id}_{S_n}$ and $q_n|_{F_n} = \pi_n$, $n = 1, 2, \dots$. One has the following commutative diagram:

$$\begin{array}{ccc} H_n & \xrightarrow{h_n} & H_{n+1} \\ \downarrow q_n & & \downarrow q_{n+1} \\ S_n \oplus T_n & \xrightarrow{\iota_n} & S_{n+1} \oplus T_{n+1} \end{array}$$

It induces a quotient map $\Pi : H \rightarrow G$. It is an order preserving map.

Now let $a_i, b_j \in G_+$, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$, as described. Let $a'_i, b'_j \in S$ such that $\Pi(a'_i) = a_i$ and $\Pi(b'_j) = b_j$, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. Then $a'_i, b'_j \in H_+$ and

$$\sum_{i \in A} a'_i \leq \sum_{j \in R_A} b'_j$$

in H_+ . Since H satisfies the assumption in 3.2, there are $c'_{ij} \in H_+$ such that

$$\sum_{j=1}^m c'_{ij} = a'_i, \quad \sum_{i=1}^n c'_{ij} = b'_j \quad \text{and} \quad (e \ 3.109)$$

$$c'_{ij} = 0 \quad \text{unless} \quad (i, j) \in R. \quad (e \ 3.110)$$

Put $c_{ij} = \Pi(c'_{ij})$, $i = 1, 2, \dots, m$ and $j = 1, 2, \dots, n$. Then

$$\sum_{j=1}^m c_{ij} = a_i, \quad \sum_{i=1}^n c_{ij} = b_j \quad \text{and} \quad (e 3.111)$$

$$c_{ij} = 0 \quad \text{unless} \quad (i, j) \in R. \quad (e 3.112)$$

□

Lemma 3.4. *Let A be a unital separable simple C^* -algebra with stable rank one and weakly unperforated $K_0(A)$ which has the Riesz interpolation property and let Ω be a compact metric space. Suppose that $\varphi_X(f) = \sum_{i=1}^m f(\xi_i)p_i$ and $\varphi_Y(f) = \sum_{j=1}^n f(\zeta_j)q_j$ for all $f \in C(\Omega)$, where $\{p_1, p_2, \dots, p_m\}$ and $\{q_1, q_2, \dots, q_n\}$ are two sets of mutually orthogonal non-zero projections in A such that $\sum_{i=1}^m p_i = \sum_{j=1}^n q_j = 1_A$ and $\xi_i, \zeta_j \in \Omega$. Let $d > 0$. Then $D_c(\varphi_X, \varphi_Y) \leq d$ if and only if, for any $\epsilon > 0$, there are projections $p_{i,j}, q_{i,j} \in A$ such that*

$$p_i = \sum_{j=1}^n p_{i,j}, \quad q_j = \sum_{i=1}^m q_{i,j}, \quad (e 3.113)$$

$$[p_{i,j}] = [q_{i,j}] \quad \text{in} \quad K_0(A) \quad \text{and} \quad (e 3.114)$$

$$\max\{\text{dist}(\xi_i, \zeta_j) : q_{i,j} \neq 0\} < d + \epsilon. \quad (e 3.115)$$

Proof. Suppose $d = D_c(\varphi_X, \varphi_Y)$. Let $\epsilon > 0$. Put

$$R = \{(i, j) : \text{dist}(\xi_i, \zeta_j) \leq d + \epsilon\}.$$

For any $A \subset \{1, \dots, m\}$, put $O_A = \{\xi_i : i \in A\}$ and $O_{R_A} = \{\zeta_j : j \in R_A\}$. Then

$$\sum_{i \in A} [p_i] = \left[\sum_{i \in A} p_i \right] = [\varphi_X(f_{O_A})] \leq [\varphi_Y(f_{(O_A)_{d+\epsilon}})] = [\varphi_Y(f_{O_{R_A}})] = \sum_{j \in R_A} [q_j].$$

It follows from 3.3, there projection r_{ij} such that

$$[p_i] = \sum_{j=1}^n [r_{ij}], \quad [q_j] = \sum_{i=1}^m [r_{ij}], \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n,$$

where $r_{ij} = 0$ unless $(i, j) \in R$. Then there are $\{p_{ij}\}$ and $\{q_{ij}\}$ with $[p_{ij}] = [q_{ij}] = [r_{ij}]$, satisfying

$$p_i = \sum_{j=1}^n p_{ij}, \quad q_j = \sum_{i=1}^m q_{ij}, \quad i = 1, 2, \dots, m; j = 1, 2, \dots, n.$$

Then

$$\max\{\text{dist}(\xi_i, \zeta_j) : q_{ij} \neq 0\} \leq d + \epsilon.$$

The converse is obvious. □

Lemma 3.5. *Let A be a unital separable simple C^* -algebra with stable rank one and weakly unperforated $K_0(A)$ which has the Riesz interpolation property and let $x, y \in A$ be two normal elements with finite spectrum. Then,*

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq D_c(x, y). \quad (e 3.116)$$

Proof. Let $\epsilon > 0$. Put $d = D_c(x, y) + \epsilon$. We assume $x = \sum_{i=1}^m \lambda_i p_i$, $y = \sum_{j=1}^n \mu_j q_j$, where $\{p_i\}_{i=1}^m$ and $\{q_j\}_{j=1}^n$ are mutually orthogonal projections with $\sum_{i=1}^m p_i$ and $\sum_{j=1}^n q_j = 1$. It follows from 3.4 that there are $\{p_{ij}\}$ and $\{q_{ij}\}$ with $[p_{ij}] = [q_{ij}] = [r_{ij}]$, satisfying

$$p_i = \sum_{j=1}^n p_{ij}, q_j = \sum_{i=1}^m q_{ij}, i = 1, 2, \dots, m; j = 1, 2, \dots, n.$$

Let $u \in U(A)$ with $u^* p_{ij} u = q_{ij}$, $i = 1, 2, \dots, m; j = 1, 2, \dots, n$.

$$\|u^* x u - y\| = \left\| \sum_{i,j} (\lambda_i - \mu_j) q_{ij} \right\| \leq \max\{|\lambda_i - \mu_j| : q_{ij} \neq 0\} \leq d.$$

□

Theorem 3.6. *Let A be a unital simple separable C^* -algebra with real rank zero, stable rank one and with weakly unperforated $K_0(A)$. Suppose that x and y are two normal elements in A with*

$$[\lambda - x] = 0 \text{ and } [\mu - y] = 0 \text{ in } K_1(A) \quad (\text{e 3.117})$$

for all $\lambda \notin \text{sp}(x)$ and for all $\mu \notin \text{sp}(y)$. Then

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq D_c(x, y). \quad (\text{e 3.118})$$

Proof. Let $\epsilon > 0$. The assumption (e 3.117) implies that $\lambda - x \in \text{Inv}_0(A)$ for all $\lambda \notin \text{sp}(x)$. It follows from [17] that, for any $\delta > 0$, there is a normal element $x_1 \in A$ with finite spectrum in $\text{sp}(x)$ such that

$$\|x - x_1\| < \min\{\delta, \epsilon/8\}. \quad (\text{e 3.119})$$

It follows from 2.17 that, for sufficiently small δ , we may assume that

$$D_c(x, x_1) < \epsilon/8. \quad (\text{e 3.120})$$

Exactly the same argument shows that there is normal element with finite spectrum in $\text{sp}(y)$ such that

$$\|y - y_1\| < \epsilon/8 \text{ and } D_c(y, y_1) < \epsilon/8. \quad (\text{e 3.121})$$

It follows that

$$D_c(x_1, y_1) < D_c(x, y) + \epsilon/4. \quad (\text{e 3.122})$$

Since x_1 and y_1 both have finite spectrum, it follows from 3.5 that there exists a unitary $u \in A$ such that

$$\|u^* x_1 u - y_1\| < D_c(x_1, y_1) + \epsilon/8 < D_c(x, y) + \epsilon/4 + \epsilon/8 = D_c(x, y) + 3\epsilon/8. \quad (\text{e 3.123})$$

It follows that

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) < \epsilon/8 + \|u^* x_1 u - y_1\| + \epsilon/8 \quad (\text{e 3.124})$$

$$< D_c(x, y) + 5\epsilon/8 \quad (\text{e 3.125})$$

for all $\epsilon > 0$.

□

Theorem 3.7. *Let A be a unital separable AF -algebra and let $x, y \in A$ be two normal elements. Then*

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq D_c(x, y).$$

Proof. Fix two normal elements $x, y \in A$. Let $\epsilon > 0$. Let $\epsilon > \eta > 0$. For any $\delta > 0$ with $\delta < \eta/4$, there exists a finite dimensional C^* -subalgebra $B \subset A$ with $1_B = 1_A$ and two elements $x', y' \in B$ such that

$$\|x - x'\| < \delta \text{ and } \|y - y'\| < \delta. \quad (\text{e 3.126})$$

It follows from [16] that, for some sufficiently small δ , there are normal elements $x_1, y_1 \in B$ such that

$$\|x' - x_1\| < \eta/4 \text{ and } \|y' - y_1\| < \eta/4. \quad (\text{e 3.127})$$

Thus

$$\|x - x_1\| < \eta/4 + \delta \text{ and } \|y - y_1\| < \eta/4 + \delta. \quad (\text{e 3.128})$$

So, by 2.17,

$$D_c(x, x_1) < \epsilon/4, \quad D_c(y, y_1) < \epsilon/4 \text{ and } D_c(x_1, y_1) < D_c(x, y) + \epsilon/2. \quad (\text{e 3.129})$$

Now x_1 and y_1 have finite spectra. We then complete the proof as in 3.6. \square

4 Maps with finite dimensional ranges

Definition 4.1. Let Ω be a compact metric space, $X, Y \subset \Omega$ be two compact subsets. Let A be a unital C^* -algebra and let $\varphi_X : C(\Omega) \rightarrow A$ and $\varphi_Y : C(\Omega) \rightarrow A$ be two homomorphisms with spectrum X and Y , respectively. Let $\{h_n : C(\Omega) \rightarrow A\}$ be a sequence of homomorphisms with finite dimensional ranges, i.e, $h_n(f) = \sum_{i=1}^{k(n)} f(\xi(i, n))p_{n,i}$ for all $f \in C(\Omega)$, where $\xi(i, n) \in X \cap Y$ and where $\{p_{n,1}, p_{n,2}, \dots, p_{n,k(n)}\}$ is a sequence of finite subsets of mutually orthogonal projections. We assume that, for any $\epsilon > 0$, there is $N \geq 1$ such that $\{\xi(1, n), \xi(2, n), \dots, \xi(k(n), n)\}$ is ϵ -dense in $X \cap Y$. Denote by $e_n = \sum_{i=1}^{k(n)} p_{n,i}$, $n = 1, 2, \dots$. Suppose that

$$\lim_{n \rightarrow \infty} \sup\{\tau(e_n) : \tau \in T(A)\} = 0. \quad (\text{e 4.130})$$

Let $\{u_n\} \subset U(A)$ be a sequence of unitaries, let $q_n = u_n^* e_n u_n$, let $\varphi_{X,n} : C(\Omega) \rightarrow (1 - e_n)A(1 - e_n)$ and let $\varphi_{Y,n} : C(\Omega) \rightarrow (1 - q_n)A(1 - q_n)$ be two unital homomorphisms with spectrum in X and Y , respectively. Suppose that

$$\lim_{n \rightarrow \infty} D_c(\varphi_X, h_n + \varphi_{X,n}) = 0 \text{ and } \lim_{n \rightarrow \infty} D_c(\varphi_Y, \text{Ad } u_n \circ h_n + \varphi_{Y,n}) = 0. \quad (\text{e 4.131})$$

Define $\varphi'_{Y,n}(f) = u_n \varphi_{Y,n}(f) u_n^*$ for all $f \in C(\Omega)$. Then $\varphi'_{Y,n} : C(\Omega) \rightarrow (1 - e_n)A(1 - e_n)$. Denote by $D_c(\varphi_{X,n}, \varphi'_{Y,n})$ the distance defined in 2.10 for $A = (1 - e_n)A(1 - e_n)$.

Now we defined

$$D_c^e(\varphi_X, \varphi_Y) = \inf\{\liminf_{n \rightarrow \infty} D_c(\varphi_{X,n}, \varphi'_{Y,n})\}, \quad (\text{e 4.132})$$

where the infimum is taken among all possible such non-zero $h_n, \varphi_{X,n}, u_n$ and $\varphi_{Y,n}$.

It is important to note that

$$D_c^e(\varphi_X, \text{Ad } u \circ \varphi_X) = 0 \quad (\text{e 4.133})$$

for any compact subset $X \subset \Omega$ and any unitary $u \in A$. Note since

$$\text{Ad} u_n^* \circ (\text{Ad} u_n \circ h_n + \varphi_{Y,n}) = h_n + \text{Ad} u_n^* \circ \varphi_{Y,n},$$

when we consider $D_c^e(\varphi_X, \varphi_Y)$ for a pair φ_X and φ_Y , we may always replace φ_Y by $\text{Ad} u_n^* \circ \varphi_Y$ by $\varphi_{Y,n}$ and $\text{Ad} u_n^* \circ \varphi_{Y,n}$ respectively, and write

$$\lim_{n \rightarrow \infty} D_c(\varphi_X, h_n + \varphi_{X,n}) = 0 \text{ and } \lim_{n \rightarrow \infty} D_c(\varphi_Y, h_n + \varphi_{Y,n}) = 0 \quad (\text{e 4.134})$$

This will be used throughout the paper without further notice.

If no such non-zero maps $\{h_n\}$ exists, we define $D_c^e(\varphi_X, \varphi_Y) = D_c(\varphi_X, \varphi_Y)$. In particular, if $X \cap Y = \emptyset$, $D_c^e(\varphi_X, \varphi_Y) = D_c(\varphi_X, \varphi_Y)$. When $X \cap Y \neq \emptyset$ and A is a unital separable simple C^* -algebra of real rank zero, stable rank one and weakly unperforated $K_0(A)$, non-zero $\{h_n\}$ can always be found (see 4.14 below). In general,

$$D_c(\varphi_X, \varphi_Y) \leq D_c^e(\varphi_X, \varphi_Y). \quad (\text{e 4.135})$$

From the definition, by 2.15,

$$D_c^e(\varphi_X, \varphi_Y) = D_c^e(\varphi_Y, \varphi_X). \quad (\text{e 4.136})$$

We also note that

$$D_c^e(\varphi_X, \varphi_Y) = \inf \{ \limsup_{n \rightarrow \infty} D_c(\varphi_{X,n}, \varphi'_{Y,n}) \}. \quad (\text{e 4.137})$$

To see this, take a subsequence $\{n_k\}$ such that

$$\lim_{k \rightarrow \infty} D_c(\varphi_{X,n_k}, \varphi'_{Y,n_k}) = \limsup_{n \rightarrow \infty} D_c(\varphi_{X,n}, \varphi'_{Y,n}).$$

Then

$$\liminf_{k \rightarrow \infty} D_c(\varphi_{X,n_k}, \varphi'_{Y,n_k}) = \lim_{k \rightarrow \infty} D_c(\varphi_{X,n_k}, \varphi'_{Y,n_k}).$$

Thus, by the definition of $D_c^e(\cdot, \cdot)$, (e 4.137) holds. Furthermore, there exists a subsequence $\{n_k\}$ such that

$$\lim_{k \rightarrow \infty} D_c(\varphi_{X,n_k}, \varphi_{Y,n_k}) = D_c^e(\varphi_X, \varphi_Y).$$

To see this, for each k , there exists such sequence $\{h_{n,k}\}$, $\varphi_{X,n,k}$ and $\varphi_{Y,n,k}$ such that

$$\liminf_{n \rightarrow \infty} D_c(\varphi_{X,n,k}, \varphi_{Y,n,k}) \leq D_c^e(\varphi_X, \varphi_Y) + 1/k.$$

Choose $\{n_k\}$ such that

$$D_c(\varphi_{X,n_k,k}, \varphi_{Y,n_k,k}) \leq D_c^e(\varphi_X, \varphi_Y) + 1/k.,$$

Then

$$\lim_{k \rightarrow \infty} D_c(\varphi_{X,n_k,k}, \varphi_{Y,n_k,k}) = D_c^e(\varphi_X, \varphi_Y).$$

Remark 4.2. Let A be a unital simple C^* -algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$. Suppose that $\varphi_X(f) = \sum_{i=1}^m f(\xi_i) p_i + \sum_{i=m+1}^{n_1} f(\zeta_i) p_i$ and $\varphi_Y(f) = \sum_{i=1}^m f(\xi_i) q_i + \sum_{i=m+1}^{n_2} f(\eta_i) q_i$ for all $f \in C(\Omega)$, where $\{p_1, p_2, \dots, p_{n_1}\}$ and $\{q_1, q_2, \dots, q_{n_2}\}$ are two sets of mutually orthogonal non-zero projections such that $\sum_{i=1}^{n_1} p_i = 1_A = \sum_{i=1}^{n_2} q_i$,

$\{\xi_1, \xi_2, \dots, \xi_m\} = X \cap Y$ and $\zeta_i \in X \setminus Y$ and $\eta_i \in Y \setminus X$. Let $e_{i,n} \leq p_i$ and $e'_{i,n} \leq q_i$ be non-zero projections such that $[e_{1,n}] = [e_{i,n}] = [e'_{i,n}]$ for all i and n , and

$$\lim_{n \rightarrow \infty} \sup \left\{ \tau \left(\sum_{i=1}^m e_{i,n} \right) : \tau \in T(A) \right\} = 0. \quad (\text{e 4.138})$$

Then

$$\varphi_X(f) = \sum_{i=1}^m f(\xi_i) e_{i,n} + \left(\sum_{i=1}^m f(\xi_i) (p_i - e_{i,n}) + \sum_{i=m+1}^{n_1} f(\zeta_i) p_i \right) \quad \text{and} \quad (\text{e 4.139})$$

$$\varphi_Y(f) = \sum_{i=1}^m f(\xi_i) e'_{i,n} + \left(\sum_{i=1}^m f(\xi_i) (q_i - e'_{i,n}) + \sum_{i=m+1}^{n_2} f(\eta_i) q_i \right) \quad (\text{e 4.140})$$

for all $f \in C(\Omega)$. It makes sense that one insists that $e_{i,n}$ pairs with $e'_{i,n}$ and the rest of them pairs according to the distance D_c defined in 2.10. After all, $e_{i,n}$ and $e'_{i,n}$ corresponds to the same point $\xi_i \in X \cap Y$.

Proposition 4.3. *Let Ω be a compact metric space and $X \subset \Omega$. Let $\Delta : (0, 1) \rightarrow (0, 1)$ be an increasing function (with $\lim_{r \rightarrow 0} \Delta(r) = 0$). For any $\epsilon > 0$, let $r_0 = \epsilon/16$. There is a finite subset of mutually orthogonal projections $\{f_1, f_2, \dots, f_n\} \subset C(\overline{X_{\epsilon/64}})$, a finite subset $\mathcal{H} \subset C(\overline{X_{\epsilon/64}})_+$ and $\sigma > 0$ satisfying the following: Suppose that A is a unital simple C^* -algebra with stable rank one and with strict comparison for positive elements and suppose that $\varphi, \psi : C(\overline{X_{\epsilon/64}}) \rightarrow A$ are two homomorphisms such that*

$$\varphi_{*0}([f_i]) = \psi_{*0}([f_i]), \quad i = 1, 2, \dots, n, \quad (\text{e 4.141})$$

$$|\tau \circ \varphi(g) - \tau \circ \psi(g)| < \sigma \quad \text{for all } g \in \mathcal{H} \quad \text{and} \quad (\text{e 4.142})$$

$$\mu_{\tau \circ \varphi}(O) \geq \Delta(r) \quad (\text{e 4.143})$$

for all open balls $O \subset \overline{X_{\epsilon/64}}$ with radius $r > r_0$ and for all $\tau \in T(A)$, then

$$D_c(\varphi, \psi) < \epsilon. \quad (\text{e 4.144})$$

Moreover, σ can be chosen to be

$$\sigma = (1/4)\Delta(\epsilon/16). \quad (\text{e 4.145})$$

Proof. Let $\epsilon > 0$. Let $\Omega_1 = \overline{X_{\epsilon/64}}$. Since Ω_1 is compact, there are $\xi_1, \xi_2, \dots, \xi_{m_1} \in \Omega_1$ such that $\bigcup_{i=1}^{m_1} O(\xi_i, \epsilon/16) \supset \Omega_1$. Let G_1, G_2, \dots, G_K be all possible finite union of those $O(\xi_i, \epsilon/16)$'s. If $O \subset \Omega_1$ is any open subset, there is G_j such that

$$O \subset G_j \quad \text{and} \quad d_H(O, G_j) \leq \epsilon/8. \quad (\text{e 4.146})$$

Without loss of generality, we may assume that $(G_1)_{5\epsilon/64}, (G_2)_{5\epsilon/64}, \dots, (G_{K_0})_{5\epsilon/64}$ are clopen sets and $(G_{K_0+1})_{5\epsilon/64}, (G_{K_0+2})_{5\epsilon/64}, \dots, (G_K)_{5\epsilon/64}$ are not closed. Therefore, for $j > K_0$, there is $\zeta'_j \in \Omega_1$ such that $\zeta'_j \notin (G_j)_{5\epsilon/64}$ and $\text{dist}(\zeta'_j, G_j) = 5\epsilon/64$. This implies there exists $\zeta_j \in X$ such that $4\epsilon/64 < \text{dist}(\zeta_j, G_j) < 6\epsilon/64$. There is $\eta > 0$ such that $\zeta_j \notin (G_j)_{\eta+\epsilon/16}$, $K_0 < j \leq K$.

Note that

$$O(\zeta_j, \epsilon/16) \subset (G_j)_{10\epsilon/64} \setminus (G_j)_\eta. \quad (\text{e 4.147})$$

Therefore

$$\inf\{d_\tau(\varphi(f_{(G_j)_{10\epsilon/64}}) - d_\tau(\varphi(f_{G_j}))) : \tau \in T(A)\} \geq \Delta(\epsilon/16). \quad (\text{e 4.148})$$

Let S_1, S_2, \dots, S_n be clopen subsets of Ω_1 such that $S_j = (G_j)_{5\epsilon/64}$, $j = 1, 2, \dots, K_0$, and $n \geq K_0$. Let f_j be the characteristic function of S_j $j = 1, 2, \dots, n$. Let $\mathcal{F} \subset C(\Omega_1)_+$ be the finite subset which contains g_j , $j = 1, 2, \dots, K$, where $0 \leq g_j \leq 1$, $g_j(t) = 1$ on $(G_j)_{10\epsilon/64}$ and $g_j(t) = 0$ if $t \notin (G_j)_{\epsilon/4}$.

Choose $0 < \sigma = (1/4)\Delta(\epsilon/16)$.

Now suppose that $\psi : C(\Omega_1) \rightarrow A$ is a unital homomorphism which satisfies the assumption (e 4.141), (e 4.142) and (e 4.143).

Let $O \subset \Omega_1$ be an open subset. Then we may assume that (e 4.146) holds. In particular,

$$(G_j)_{\epsilon/4} \subset O_\epsilon. \quad (\text{e 4.149})$$

If $1 \leq j \leq K_0$, by the assumption (e 4.141),

$$[\varphi(f_{(G_j)_{5\epsilon/64}})] = [\psi(f_{(G_j)_{5\epsilon/64}})], \quad j = 1, 2, \dots, K_0. \quad (\text{e 4.150})$$

Therefore

$$\varphi(f_O) \lesssim \varphi(f_{(G_j)_{5\epsilon/64}}) \lesssim \psi(f_{(G_j)_{5\epsilon/64}}) \lesssim \psi(f_{(G_j)_{\epsilon/4}}) \lesssim \psi(f_{O_\epsilon}). \quad (\text{e 4.151})$$

If $K_0 < j \leq K$,

$$d_\tau(\psi(f_{O_\epsilon})) \geq \tau(\psi(g_j)) > \tau(\varphi(g_j)) - \sigma \quad (\text{e 4.152})$$

$$\geq d_\tau(\varphi(g_{(G_j)_{10\epsilon/64}})) - \sigma \quad (\text{e 4.153})$$

$$> d_\tau(\varphi(g_{G_j})) \geq d_\tau(\varphi(f_O)) \quad (\text{e 4.154})$$

for all $\tau \in T(A)$. Combining (e 4.151) and (e 4.152), since A has the strict comparison for positive elements, we obtain

$$D_c(\varphi, \psi) < \epsilon. \quad (\text{e 4.155})$$

□

Corollary 4.4. *Let $\epsilon > 0$. Let X be a compact subset of the plane. Suppose that $X = \sqcup_{i=1}^n S_i$, where each S_i is an $\epsilon/64$ -connected component, $i = 1, 2, \dots, n$, suppose that A is a unital simple C^* -algebra of stable rank one, real rank zero and weakly unperforated $K_0(A)$. Suppose that $\{\lambda_1, \lambda_2, \dots, \lambda_{m_0}\}$, $\{\mu_1, \mu_2, \dots, \mu_{m_1}\}$ and $\{\zeta_1, \zeta_2, \dots, \zeta_{m_2}\}$ are $\epsilon/16$ -dense in X , and suppose that $\{e_{0,1}, e_{0,2}, \dots, e_{0,m_0}\}$, $\{e_{1,1}, e_{1,2}, \dots, e_{1,m_1}\}$ and $\{e_{2,1}, e_{2,2}, \dots, e_{2,m_2}\}$ are mutually orthogonal non-zero projections in A such that*

$$P = \sum_{j=1}^{m_1} e_{1,j} = \sum_{j=1}^{m_2} e_{2,j}, \quad (\text{e 4.156})$$

$$e_{0,j} \in (1 - P)A(1 - P), \quad j = 1, 2, \dots, m_0, \quad (\text{e 4.157})$$

$$8\tau(P) < \tau(e_{0,j}) \text{ for all } \tau \in T(A), \quad j = 1, 2, \dots, m_0 \text{ and} \quad (\text{e 4.158})$$

$$\varphi_{*0}([\chi_{S_i}]) = \psi_{*0}([\chi_{S_i}]), \quad i = 1, 2, \dots, n, \quad (\text{e 4.159})$$

where $\varphi(f) = \sum_{j=1}^{m_0} f(\lambda_j)e_{0,j} + \sum_{j=1}^{m_1} f(\mu_j)e_{1,j}$ and $\psi(f) = \sum_{j=1}^{m_0} f(\lambda_j)e_{0,j} + \sum_{j=1}^{m_2} f(\zeta_j)e_{2,j}$. Then there is a unitary $u \in A$ such that

$$\|u^*\varphi(z)u - \psi(z)\| < \epsilon, \quad (\text{e 4.160})$$

where $z \in C(X)$ is the identity function on X .

Proof. The proof is virtually contained in that of 4.3. We can keep all notations and proof of 4.3 up to the definition of σ .

We define

$$\sigma = \min\{\inf\{\tau(e_{0,j}) : \tau \in T(A)\} : j = 1, 2, \dots, m_0\}.$$

Since A is unital and simple, $\sigma > 0$. By the proof of 4.3, for any O , there is G_j such that

$$O \subset G_j \subset (G_j)_{\epsilon/8} \subset (G_j)_{5\epsilon/64} = S_j.$$

If $O_{\epsilon/16} = S_j$, then

$$\psi(f_O) \lesssim \psi(\chi_{S_j}) \lesssim \varphi(\chi_{S_j}) \lesssim \varphi(f_{O_{\epsilon/16}}). \quad (\text{e 4.161})$$

In general,

$$d_\tau(\psi(f_O)) < d_\tau(\varphi(f_O)) + (1/4)\sigma \quad (\text{e 4.162})$$

for all $\tau \in T(A)$. If $O_{\epsilon/16} \neq S_j$, there is $\xi \notin O_{\epsilon/16}$ such that $\text{dist}(\xi, O_{\epsilon/16}) < \epsilon/64$. There is λ_j such that

$$\text{dist}(\xi, \lambda_j) < \epsilon/16. \quad (\text{e 4.163})$$

Then

$$\lambda_j \notin O \text{ and } \lambda_j \in O_{\epsilon/8}. \quad (\text{e 4.164})$$

Then, by (e 4.162),

$$d_\tau(\psi(f_O)) \leq d_\tau(\varphi(f_O)) + (1/4)\sigma \quad (\text{e 4.165})$$

$$< d_\tau(\varphi(f_O)) + d_\tau(\varphi(O(\lambda_j, \epsilon/16))) \leq d_\tau(\varphi(f_{O_\epsilon})). \quad (\text{e 4.166})$$

It follows that

$$\psi(f_O) \lesssim \varphi(f_{O_\epsilon}). \quad (\text{e 4.167})$$

This holds for any open set $O \subset X$. Therefore

$$D_c(\psi(z), \varphi(z)) < \epsilon. \quad (\text{e 4.168})$$

The corollary then follows from 3.5. □

The following follows from Proposition 11.1 of [23] immediately.

Proposition 4.5. *Let A be a unital simple C^* -algebra with $T(A) \neq \emptyset$ and let X be a compact metric space. Suppose that $\varphi : C(X) \rightarrow A$ is a unital monomorphism. Then there is an increasing function $\Delta : (0, 1) \rightarrow (0, 1)$ with $\lim_{r \rightarrow 0} \Delta(r) = 0$ such that*

$$\mu_{\tau \circ \varphi}(O) \geq \Delta(r) \text{ for all } \tau \in T(A) \quad (\text{e 4.169})$$

where O is an open ball of X with radius $r \in (0, 1)$.

Corollary 4.6. *Let A be a unital simple C^* -algebra with stable rank one and with strict comparison for positive elements, let Ω be a compact metric space, let $X \subset \Omega$ be a compact subset and let $\varphi_X : C(\Omega) \rightarrow A$ be a unital homomorphism with spectrum X . For any $\epsilon > 0$, there is $\delta > 0$, a finite set of clopen subsets S_1, S_2, \dots, S_k in $\overline{X_{\epsilon/64}}$ and a finite subset $\mathcal{F} \subset C(\overline{X_{\epsilon/64}})_+$ such that for any unital homomorphism $\varphi_Y : C(\overline{X_{\epsilon/64}}) \rightarrow A$ with the property that*

$$|\tau(\varphi_X(f)) - \tau(\varphi_Y(f))| < \delta \text{ for all } f \in \mathcal{F} \text{ and} \quad (\text{e 4.170})$$

$$[\varphi_X(\chi_{S_i})] = [\varphi_Y(\chi_{S_i})], \quad i = 1, 2, \dots, k, \quad (\text{e 4.171})$$

then

$$D_c(\varphi_X, \varphi_Y) < \epsilon. \quad (\text{e 4.172})$$

Proposition 4.7. *Let Ω be a compact metric space, let $X \subset \Omega$ be a compact subset, let A be a unital simple C^* -algebra of stable rank one and with strict comparison for positive elements and let $\varphi_X : C(\Omega) \rightarrow A$ be a unital homomorphism with spectrum X . Suppose that $\{h_n : C(\Omega) \rightarrow A\}$ is a sequence of unital homomorphisms such that*

$$\lim_{n \rightarrow \infty} D_c(\varphi_X, h_n) = 0. \quad (\text{e 4.173})$$

Then

$$\lim_{n \rightarrow \infty} D_c^e(\varphi_X, h_n) = 0. \quad (\text{e 4.174})$$

Proof. Fix φ_X . Let $\epsilon > 0$. Without loss of generality, we may assume that $\Omega = \overline{X_{\epsilon/64}}$. Let S_1, S_2, \dots, S_k be a finite set of clopen subsets of Ω , $\mathcal{F} \subset C(\Omega)_+$ be a finite subset and $\delta > 0$ be required by 4.6 for φ_X and ϵ . Let Y_n be the spectrum of h_n . By (e 4.173), we assume that $Y_n \subset \overline{X_{\epsilon/64}}$ for all n , without loss of generality. Furthermore, we may further assume that $X \cap Y_n \neq \emptyset$ for all n . Suppose that $\{e_n\} \subset A$ is a sequence of projections, $\{\varphi_{0,n} : C(\Omega) \rightarrow e_n A e_n\}$ is a sequence of unital homomorphisms with finite spectrum that is ϵ_n -dense in $X \cap Y_n$ and with $\lim_{n \rightarrow \infty} \epsilon_n = 0$, and two sequences of unital homomorphisms $\{\varphi_{1,n}, \varphi_{2,n} : C(\Omega) \rightarrow (1 - e_n)A(1 - e_n)\}$ such that the spectrum of $\varphi_{1,n}$ is in X , the spectrum of $\varphi_{2,n}$ is in $X \cap Y_n$, $n = 1, 2, \dots$ and

$$\lim_{n \rightarrow \infty} \sup\{\tau(e_n) : \tau \in T(A)\} = 0, \quad (\text{e 4.175})$$

$$\lim_{n \rightarrow \infty} D_c(\varphi_X, \varphi_{0,n} + \varphi_{1,n}) = 0 \text{ and} \quad (\text{e 4.176})$$

$$\lim_{n \rightarrow \infty} D_c(h_n, \varphi_{0,n} + \varphi_{2,n}) = 0. \quad (\text{e 4.177})$$

By 2.16, we may assume that, for all $n \geq 1$,

$$[\varphi_X(\chi_{S_i})] = [\varphi_{0,n}(\chi_{S_i})] + [\varphi_{1,n}(\chi_{S_i})] = [\varphi_{0,n}(\chi_{S_i})] + [\varphi_{2,n}(\chi_{S_i})], \quad (\text{e 4.178})$$

$j = 1, 2, \dots, k$ and $n = 1, 2, \dots$. Since A has stable rank one, this implies that

$$[\varphi_{1,n}(\chi_{S_i})] = [\varphi_{2,n}(\chi_{S_i})], \quad j = 1, 2, \dots, k \text{ and } n = 1, 2, \dots \quad (\text{e 4.179})$$

By (e 4.175), there exists $N \geq 1$ such that, for all $n \geq N$,

$$\sup\{\tau(e_n) : \tau \in T(A)\} < \delta. \quad (\text{e 4.180})$$

It follows from (e 4.176), (e 4.177) and (e 4.173) that

$$\sup\{|\tau(\varphi_{1,n}(f)) - \tau(\varphi_{2,n}(f))| : \tau \in T(A)\} < \delta \text{ for all } f \in \mathcal{F}. \quad (\text{e 4.181})$$

It follows from 4.6 that

$$D_c(\varphi_{1,n}, \varphi_{2,n}) < \epsilon \text{ for all } n \geq N. \quad (\text{e 4.182})$$

It follows that

$$\limsup_{n \rightarrow \infty} D_c^e(\varphi_X, h_n) = 0. \quad (\text{e 4.183})$$

□

In fact, we proved the following:

Corollary 4.8. *With the same assumption above, for a fixed unital homomorphism $\varphi : C(\Omega) \rightarrow A$, and for any $\epsilon > 0$, there exists $\delta > 0$ such that if $\psi : C(\Omega) \rightarrow A$ is another unital homomorphism such that $D_c(\varphi, \psi) < \delta$, then*

$$D_c^e(\varphi, \psi) < \epsilon. \quad (\text{e 4.184})$$

Since for any φ , $D_c(\varphi, \varphi) = 0$, we have

$$D_c^e(\varphi, \varphi) = 0. \quad (\text{e 4.185})$$

We also have the following:

Proposition 4.9. *Let A be a unital simple C^* -algebra with strict comparison for positive elements. Then*

$$D_c^e(\varphi_X, \varphi_Y) \leq D^T(\varphi_X, \varphi_Y). \quad (\text{e 4.186})$$

Proof. By (e 2.55) in 2.21 and (e 4.135), we may assume $X \cap Y \neq \emptyset$ and let $\Omega = X \cup Y$. Suppose that $\{e_n\} \subset A$ is a sequence of projections, $h_n : C(\Omega) \rightarrow e_n A e_n$ is a sequence of unital homomorphisms with spectrum being ϵ_n -dense in $X \cap Y$ and with $\lim_{n \rightarrow \infty} \epsilon_n = 0$, and two sequences of unital homomorphisms $\varphi_{X_n}, \varphi_{Y_n} : C(\Omega) \rightarrow (1 - e_n)A(1 - e_n)$ such that $X_n \subset X$, $n = 1, 2, \dots$ and

$$\lim_{n \rightarrow \infty} \sup\{\tau(e_n) : \tau \in T(A)\} = 0, \quad (\text{e 4.187})$$

$$\lim_{n \rightarrow \infty} D_c(\varphi_X, \varphi_{X_n} + h_n) = 0 \text{ and} \quad (\text{e 4.188})$$

$$\lim_{n \rightarrow \infty} D_c(\varphi_Y, \varphi_{Y_n} + h_n) = 0 \quad (\text{e 4.189})$$

(see the lines around (e 4.134)).

For any fixed $\epsilon > 0$, let $r_0 = D^T(\varphi_X, \varphi_Y)$. We will show $D_c^e(\varphi_X, \varphi_Y) < r_0 + 3\epsilon$. It suffices to show there exists integer N such that for all $n > N$,

$$D_c(\varphi_{X_n}, \varphi_{Y_n}) < r_0 + 3\epsilon. \quad (\text{e 4.190})$$

There are finitely many open subsets $\{O_k : k = 1, 2, \dots, K\}$ of Ω such that for any open set $O \subset \Omega$, there exists O_k such that

$$O \subset O_k \subset O_\epsilon. \quad (\text{e 4.191})$$

We may assume that

$$O_k \cap X \neq X, k = 1, 2, \dots, J; O_k \cap X = X, k = J + 1, J + 2, \dots, K.$$

Define

$$\delta = \inf\{d_\tau(\varphi_Y(f_{(O_k)_{r_0+\epsilon}})) - d_\tau(\varphi_X(f_{O_k})) : \tau \in T(A), k = 1, 2, \dots, J\}. \quad (\text{e 4.192})$$

By 2.20, $\delta > 0$.

Let N be large such that for any $n > N$,

$$\sup\{\tau(e_n) : \tau \in T(A)\} < \frac{\delta}{2} \quad (\text{e 4.193})$$

and

$$D_c(\varphi_X, \varphi_{X_n} + h_n) < \epsilon, \quad D_c(\varphi_Y, \varphi_{Y_n} + h_n) < \epsilon. \quad (\text{e 4.194})$$

Since $D_T(\varphi_X, \varphi_{X_n} + h_n) \leq D_c(\varphi_X, \varphi_{X_n} + h_n) < \epsilon$, for any open subset O with $O_\epsilon \cap X \neq X$, we have

$$d_\tau(\varphi_{X_n}(f_O)) \leq d_\tau(\varphi_{X_n}(f_O)) + d_\tau(h_n(f_O)) \leq d_\tau(\varphi_X(f_{O_\epsilon})). \quad (\text{e 4.195})$$

Suppose $O_\epsilon \subset O_k \subset (O_\epsilon)_\epsilon$, then $k \leq J$ and by (e 4.192),

$$d_\tau(\varphi_X(f_{O_\epsilon})) \leq d_\tau(\varphi_X(f_{O_k})) < d_\tau(\varphi_Y(f_{(O_k)_{r_0+\epsilon}})) - \frac{\delta}{2}. \quad (\text{e 4.196})$$

Using the fact that $D_c(\varphi_X, \varphi_{X_n} + h_n) < \epsilon$, we also have

$$d_\tau(\varphi_Y(f_{(O_k)_{r_0+\epsilon}})) \leq d_\tau(\varphi_{Y_n}(f_{(O_k)_{r_0+2\epsilon}})) + d_\tau(h_n(f_{(O_k)_{r_0+2\epsilon}})) \quad (\text{e 4.197})$$

and

$$d_\tau(\varphi_{Y_n}(f_{(O_k)_{r_0+2\epsilon}})) \leq d_\tau(\varphi_{Y_n}(f_{O_{r_0+3\epsilon}})) \quad (\text{e 4.198})$$

Further since $d_\tau(h_n(f_{(O_k)_{r_0+2\epsilon}})) \leq \tau(e_n) < \frac{\delta}{2}$, combining (e 4.195), (e 4.196), (e 4.197) and (e 4.198), we obtain

$$d_\tau(\varphi_{X_n}(f_O)) < d_\tau(\varphi_{Y_n}(f_{O_{r_0+3\epsilon}})). \quad (\text{e 4.199})$$

The strict comparison for positive elements implies $[\varphi_{X_n}(f_O)] \leq [\varphi_{Y_n}(f_{O_{r_0+3\epsilon}})]$.

By the exactly the same argument, $[\varphi_{Y_n}(f_O)] \leq [\varphi_{X_n}(f_{O_{r_0+3\epsilon}})]$.

Since $D^T(\varphi_X, \varphi_Y) \geq d_H(X, Y)$, for any open set O with $O_\epsilon \cap X = X$, we have $O_{r_0+3\epsilon} \cap Y = Y$, this implies

$$[\varphi_{X_n}(f_O)] \leq [\varphi_{Y_n}(f_{O_{r_0+3\epsilon}})].$$

Therefore $D_c(\varphi_{X_n}, \varphi_{Y_n}) \leq r_0 + 3\epsilon$ holds for all $n > N$. That is the end of the proof. \square

Corollary 4.10. *Let A be a unital simple C^* -algebra with strict comparison for positive elements and with $T(A) \neq \emptyset$. Suppose that X , or Y are connected compact subsets. Then*

$$D_c(\varphi_X, \varphi_Y) = D_c^e(\varphi_X, \varphi_Y). \quad (\text{e 4.200})$$

Proof. This follows from 4.9 and the last part of 2.21. \square

Proposition 4.11. *Let A be a unital simple C^* -algebra with stable rank one, with the strict comparison for positive elements. Suppose that $\varphi_X, \varphi_Y : C(X \cup Y) \rightarrow A$ are two unital homomorphisms with spectrum X and Y , respectively. Let $\{\xi_1, \xi_2, \dots, \xi_k\} \subset X \cap Y$. Suppose that there is a sequence of finite subsets of mutually orthogonal non-zero projections $\{e_{1,n}, e_{2,n}, \dots, e_{k,n}\}$ of A such that*

$$\lim_{n \rightarrow \infty} D_c(\varphi_{X,n}, \varphi_X) = 0, \quad \lim_{n \rightarrow \infty} D_c(\varphi_{Y,n}, \varphi_Y) = 0, \quad (\text{e 4.201})$$

$$\text{and } \lim_{n \rightarrow \infty} \sup \left\{ \sum_{i=1}^k \tau(e_{i,n}) : \tau \in T(A) \right\} = 0, \quad (\text{e 4.202})$$

where $\varphi_{X,n}(f) = \sum_{i=1}^k f(\xi_i) e_{i,n} + \psi_{X,n}(f)$, $\varphi_{Y,n}(f) = \sum_{i=1}^k f(\xi_i) e_{i,n} + \psi_{Y,n}(f)$ for all $f \in C(X \cup Y)$, and $\psi_{X,n}, \psi_{Y,n} : C(X \cup Y) \rightarrow (1 - p_n)A(1 - p_n)$ are unital homomorphisms with the spectrum in X and Y , respectively, where $p_n = \sum_{i=1}^k e_{i,n}$ (please see lines around (e 4.134)).

Then

$$\liminf_{n \rightarrow \infty} D_c(\psi_{X,n}, \psi_{Y,n}) \leq D_c^e(\varphi_X, \varphi_Y). \quad (\text{e 4.203})$$

Proof. Let $d = D_c^e(\varphi_X, \varphi_Y)$ and let $\Omega = X \cup Y$. Let $\{\xi_1^{(n)}, \xi_2^{(n)}, \dots, \xi_{k(n)}^{(n)}\}$ be a sequence of finite subsets of $X \cap Y$ which are ϵ_n -dense with $\lim_{n \rightarrow \infty} \epsilon_n = 0$, let $\{e_1^{(n)}, e_2^{(n)}, \dots, e_{k(n)}^{(n)}\}$ be a sequence of mutually orthogonal non-zero projections in A with

$$\lim_{n \rightarrow \infty} \sup \left\{ \sum_{i=1}^{k(n)} \tau(e_i^{(n)}) : \tau \in T(A) \right\} = 0, \quad (\text{e 4.204})$$

and $\{u_n\}$ be a sequence of unitary such that

$$\lim_{n \rightarrow \infty} D_c(\varphi_X, h_{X,n,0} + h_{X,n,1}) = 0, \quad \lim_{n \rightarrow \infty} D_c(\varphi_Y, h_{Y,n,0} + h_{Y,n,1}) = 0 \quad \text{and} \quad (\text{e 4.205})$$

$$\lim_{n \rightarrow \infty} D_c(h_{X,n,1}, h_{Y,n,1}) = d = D_c^e(\varphi_X, \varphi_Y), \quad (\text{e 4.206})$$

where $h_{X,n,0}(f) = h_{Y,n,0}(f) = \sum_{i=1}^{k(n)} f(\xi_i^{(n)}) e_i^{(n)}$ for all $f \in C(\Omega)$ and $h_{X,n,1}, h_{Y,n,1} : C(\Omega) \rightarrow (1 - E_n)A(1 - E_n)$ are unital homomorphisms with spectrum in X and Y , respectively, and where $E_n = \sum_{i=1}^{k(n)} e_i^{(n)}$, $n = 1, 2, \dots$. Without loss of generality, we may also assume that $k(n) \geq k$ and $\xi_i^{(n)} = \xi_i$, $i = 1, 2, \dots, k$. By (e 4.202), since A has strict comparison, by passing to a subsequence of $\varphi_{i,n}$ ($\psi_{X,n}$ and $\psi_{Y,n}$), if necessary, we may further assume that

$$e_{j,n} \leq e_j^{(n)}, \quad j = 1, 2, \dots, k \quad \text{and} \quad n = 1, 2, \dots \quad (\text{e 4.207})$$

Define

$$h'_{X,n,0}(f) = \sum_{i=1}^k f(\xi_i^{(n)}) e_{i,n}, \quad (\text{e 4.208})$$

$$h''_{X,n,0}(f) = \sum_{i=1}^k f(\xi_i^{(n)}) (e_i^{(n)} - e_{i,n}) + \sum_{i=k+1}^{k(n)} f(\xi_i^{(n)}) e_i^{(n)} \quad \text{for all } f \in C(\Omega) \quad (\text{e 4.209})$$

$$h'_{X,n,1} = h''_{X,n,0} + h_{X,n,1} \quad \text{and} \quad h'_{Y,n,1} = h''_{X,n,0} + h_{Y,n,1}. \quad (\text{e 4.210})$$

Then, by assumption,

$$\lim_{n \rightarrow \infty} D_c(h'_{X,n,0} + \psi_{X,n}, h'_{X,n,0} + h'_{X,n,1}) = 0 \quad (\text{e 4.211})$$

and

$$\lim_{n \rightarrow \infty} D_c(h'_{X,n,0} + \psi_{Y,n}, h'_{X,n,0} + h_{Y,n,1}) = 0. \quad (\text{e 4.212})$$

By the proof of 4.7, (e 4.211) and (e 4.212) imply that

$$\lim_{n \rightarrow \infty} D_c(\psi_{X,n}, h'_{X,n,1}) = 0 \quad \text{and} \quad \lim_{n \rightarrow \infty} D_c(\psi_{Y,n}, h'_{Y,n,1}) = 0. \quad (\text{e 4.213})$$

It follows that

$$\liminf_{n \rightarrow \infty} D_c(\psi_{X,n}, \psi_{Y,n}) \leq \lim_{n \rightarrow \infty} D_c(\psi_{X,n}, h'_{X,n,1}) + \quad (\text{e 4.214})$$

$$\limsup_{n \rightarrow \infty} D_c(h'_{X,n,1}, h'_{Y,n,1}) + \lim_{n \rightarrow \infty} D_c(\psi_{Y,n}, h'_{Y,n,1}) \quad (\text{e 4.215})$$

$$= \limsup_{n \rightarrow \infty} D_c(h'_{X,n,1}, h'_{Y,n,1}) \quad (\text{e 4.216})$$

$$\leq \limsup_{n \rightarrow \infty} D_c(h_{X,n,1}, h_{Y,n,1}) = d. \quad (\text{e 4.217})$$

□

Lemma 4.12. *Let A be a unital separable simple C^* -algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$. Let $\varphi_X : C(X) \rightarrow A$ be a unital homomorphism. Then, for any $\epsilon > 0$, any $\sigma > 0$, any $\eta > 0$ and any finite η -dense subset $\{\xi_1, \xi_2, \dots, \xi_m\} \subset X$, there is a projection $e \in A$ with $\tau(e) < \sigma$ for all $\tau \in T(A)$, a unital homomorphism $\varphi_0 : C(X) \rightarrow eAe$ with spectrum $\{\xi_1, \xi_2, \dots, \xi_m\}$ and a unital homomorphism $\varphi_1 : C(X) \rightarrow (1 - e)A(1 - e)$ with finite spectrum such that*

$$D_c(\varphi_X, \varphi_0 + \varphi_1) < \epsilon. \quad (\text{e 4.218})$$

Proof. Since A is simple and has real rank zero and stable rank one with weakly unperforated $K_0(A)$, $K_0(A)$ has Riesz interpolation property by a theorem of Zhang ([33]). Moreover $\rho_A(K_0(A))$ is dense in $\text{Aff}(T(A))$. By [10], there exists a unital simple AH-algebra of no dimension growth B of real rank zero (therefore $TR(B) = 0$ –see [20]) such that

$$(K_0(B), K_0(B)_+, [1_B]) = (K_0(A), K_0(A)_+, [1_A]) \quad \text{and} \quad (\text{e 4.219})$$

$$K_1(B) = \{0\}. \quad (\text{e 4.220})$$

It follows from [19] that there exists a unital homomorphism $h : B \rightarrow A$ such that h_{*0} gives the identity in (e 4.219).

It follows from [22] that there exist unital monomorphisms $\psi'_X : C(X) \rightarrow B$ such that

$$(h \circ \psi'_X)_{*0} = (\varphi_X)_{*0} \quad \text{and} \quad \tau \circ h \circ \psi'_X = \tau \circ \varphi_X \quad (\text{e 4.221})$$

for all $\tau \in T(A)$. Define $\psi_X = h \circ \psi'_X$. Then

$$(\psi_X)_{*0} = (\varphi_X)_{*0} \quad \text{and} \quad \tau \circ \psi_X = \tau \circ \varphi_X \quad (\text{e 4.222})$$

for all $\tau \in T(A)$. These, in particular, by 4.6, imply that

$$D_c(\varphi_X, \psi_X) = 0. \quad (\text{e 4.223})$$

So, without loss of generality, we may assume now that $A = B$. In particular, B has tracial rank zero.

Let $\epsilon > 0$. Let $\delta > 0$ be a positive number, S_1, S_2, \dots, S_k be a finite set of mutually disjoint clopen subsets of X and let $\mathcal{F} \subset C(X)_+$ be a finite subset required by 4.6 for $\epsilon > 0$ and φ_X . We may assume that $X = \sqcup_{i=1}^m S_i$ and $1_{C(X)} \in \mathcal{F}$. By Lemma 4.3 of [22], there is a projection $p \neq 1_A$, a unital homomorphism $h : C(X) \rightarrow pAp$ with finite spectrum such that

$$|\tau \circ h(f) - \tau \circ \varphi_X(f)| < \delta/2 \text{ for all } f \in \mathcal{F} \text{ and} \quad (\text{e 4.224})$$

$$\tau \circ h(\chi_{S_i}) < \tau \circ \varphi_X(\chi_{S_i}), \quad i = 1, 2, \dots, k, \quad (\text{e 4.225})$$

for all $\tau \in T(A)$, and

$$h(f) = \sum_{i=1}^m f(\xi_i) e_i + h_1(f) \text{ for all } f \in C(X), \quad (\text{e 4.226})$$

where $\{e_1, e_2, \dots, e_m\} \subset pAp$ is a set of mutually orthogonal non-zero projections and $h_1 : C(X) \rightarrow (p - \sum_{i=1}^m e_i)A(p - \sum_{i=1}^m e_i)$ is a unital homomorphism with finite spectrum in X . Note that (e 4.224) implies that

$$\tau(1 - p) < \delta/2 \text{ for all } \tau \in T(A). \quad (\text{e 4.227})$$

By (e 4.225), there are mutually orthogonal projections $q_1, q_2, \dots, q_k \in (1 - p)A(1 - p)$ such that $[\varphi_X(\chi_{S_i})] = [q_i] + [h(\chi_{S_i})]$, $i = 1, 2, \dots, k$. Since $\sum_{i=1}^k \chi_{S_i} = 1_{C(X)}$ and φ_X is unital, $p + \sum_{i=1}^k q_i = 1_A$. Define $\psi_X : C(X) \rightarrow A$ by $\psi_X(f) = \sum_{j=1}^k f(\zeta_j) q_j + h(f)$ for all $f \in C(X)$, where $\zeta_j \in S_j$ is a point, $j = 1, 2, \dots, k$. We compute that,

$$[\psi_X(\chi_{S_i})] = [\varphi_X(\chi_{S_i})], \quad i = 1, 2, \dots, k. \quad (\text{e 4.228})$$

Moreover, by (e 4.227) and (e 4.224),

$$|\tau(\varphi_X(f)) - \tau(\psi_X(f))| < \delta \text{ for all } \tau \in T(A). \quad (\text{e 4.229})$$

It follows from 4.6 that

$$D_c(\varphi_X, \psi_X) < \epsilon. \quad (\text{e 4.230})$$

Since A is simple and has (SP), we can find non-zero projections $e'_i \leq e_i$ such that $\sum_{i=1}^m \tau(e'_i) < \sigma$. Put $e = \sum_{i=1}^m e'_i$. Define $\varphi_0(f) = \sum_{i=1}^m f(\xi_i) e_i$ for all $f \in C(X)$ and defined

$$\varphi_1(f) = \sum_{j=1}^k f(\zeta_j) q_j + \sum_{i=1}^m f(\xi_i) (e_i - e'_i) + h_1(f) \quad (\text{e 4.231})$$

for all $f \in C(X)$. Note that $\varphi_0 + \varphi_1 = \psi_X$. Lemma follows. \square

Corollary 4.13. *Let A be a unital separable simple C^* -algebra with real rank zero, stable rank one and with weakly unperforated $K_0(A)$ and let X be a compact metric space. Suppose that $\varphi_X : C(X) \rightarrow A$ is a unital homomorphism. Then, there exists a sequence of unital homomorphisms $\varphi_n : C(X) \rightarrow A$ with finite dimensional range such that*

$$\lim_{n \rightarrow \infty} D_c^e(\varphi_X, \varphi_n) = 0. \quad (\text{e 4.232})$$

Remark 4.14. In the case that A has real rank zero, stable rank one and weakly unperforated $K_0(A)$, 4.12 shows that, in the definition of $D_c^e(\varphi_X, \varphi_Y)$, if $X \cap Y \neq \emptyset$, we can also assume that the sequence of non-zero $\{h_n\}$ exists.

Proposition 4.15. *Let Ω be a compact metric space, let A be a unital simple C^* -algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$, and let $\varphi_X, \varphi_Y, \varphi_Z : C(\Omega) \rightarrow A$ be unital homomorphisms with spectrum X, Y and Z , respectively. If, in addition, $X \cap Y \subset Z$*

$$D_c^e(\varphi_X, \varphi_Y) \leq D_c^e(\varphi_X, \varphi_Z) + D_c^e(\varphi_Z, \varphi_Y). \quad (\text{e 4.233})$$

Proof. If $X \cap Y = \emptyset$, then

$$D_c^e(\varphi_X, \varphi_Y) = D_c(\varphi_X, \varphi_Y) \leq D_c(\varphi_X, \varphi_Z) + D_c(\varphi_Z, \varphi_Y) \leq D_c^e(\varphi_X, \varphi_Z) + D_c^e(\varphi_Z, \varphi_Y).$$

So, we assume that $X \cap Y \neq \emptyset$.

By the definition and from 4.14 above, we have nonzero sequences of projections $\{e(n, i)\}$ of A , unital homomorphisms $h(n, i) : C(\Omega) \rightarrow e(n, i)Ae(n, i)$ and unital homomorphisms $\varphi(n, i), \varphi(Z, n, i) : C(\Omega) \rightarrow (1 - e(n, i))A(1 - e(n, i))$ such that

$$\lim_{n \rightarrow \infty} \sup_{\tau \in T(A)} \tau(e(n, i)) = 0 \quad (\text{e 4.234})$$

$$\lim_{n \rightarrow \infty} D_c(\varphi_X, h(n, 1) + \varphi(n, 1)) = 0, \quad \lim_{n \rightarrow \infty} D_c(\varphi_Y, h(n, 2) + \varphi(n, 2)) = 0; \quad (\text{e 4.235})$$

$$\lim_{n \rightarrow \infty} D_c(\varphi_Z, h(n, i) + \varphi(Z, n, i)) = 0; \quad (\text{e 4.236})$$

$$D_c^e(\varphi_X, \varphi_Z) = \lim_{n \rightarrow \infty} D_c(\varphi(n, 1), \varphi(Z, n, 1)); \quad (\text{e 4.237})$$

$$D_c^e(\varphi_Y, \varphi_Z) = \lim_{n \rightarrow \infty} D_c(\varphi(n, 2), \varphi(Z, n, 2)) \quad \text{and} \quad (\text{e 4.238})$$

$$\lim_{n \rightarrow \infty} D_c(h(n, 1) + \varphi(Z, n, 1), h(n, 2) + \varphi(Z, n, 2)) = 0, \quad (\text{e 4.239})$$

the spectrum of $h(n, 1)$ is X'_n and the spectrum of $h(n, 2)$ is Y'_n , that of $\varphi(n, 1)$ is in X , that of $\varphi(n, 2)$ is in Y , $\varphi(Z, n, i)$ is in Z , where X'_n is a finite subset of $X \cap Z$ and Y'_n is a finite subset of $Z \cap Y$ which are ϵ_n -dense in $X \cap Z$ and in $Y \cap Z$ with $\lim_{n \rightarrow \infty} \epsilon_n = 0$. Since $X \cap Y \subset Z$, we may assume, without loss of generality, that $X'_n \cap Y'_n$ is ϵ_n -dense in $X \cap Y$. We write

$$h(n, i)(f) = \sum_{j=1}^{r(n, i)} f(\zeta(n, j, i)) q(n, j, i) \quad \text{for all } f \in C(\Omega), \quad (\text{e 4.240})$$

where $\{\zeta(n, 1, 1), \zeta(n, 2, 1), \dots, \zeta(n, r(n, 1), 1)\} = X'_n$, $\{\zeta(n, 1, 2), \zeta(n, 2, 2), \dots, \zeta(n, r(n, 2), 2)\} = Y'_n$ and $\{q(n, 1, i), q(n, 2, i), \dots, q(n, r(n, i), i)\}$ is a set of mutually orthogonal non-zero projections. We may further assume that

$$\zeta(n, j, 1) = \zeta(n, j, 2), \quad j = 1, 2, \dots, k(n) \leq r(n, 1), r(n, 2), \quad (\text{e 4.241})$$

where $\{\zeta(n, 1, 1), \zeta(n, 2, 1), \dots, \zeta(n, k(n), 1)\}$ is ϵ_n -dense in $X \cap Y$. Let X_n be the spectrum of $\varphi(n, 1)$ and Y_n be the spectrum of $\varphi(n, 2)$, $n = 1, 2, \dots$. Without loss of generality, we may assume that $X'_n \subset X_n$ and $Y'_n \subset Y_n$, $n = 1, 2, \dots$. Note that, without changing the sums $h(n, i) + \varphi(n, i)$, $h(n, i) + \varphi(Z, n, i)$ and (e 4.234)–(e 4.238), one can choose smaller $q(n, j, i)$, $j = 1, 2, \dots, r(n, i)$, $i = 1, 2$ and $n = 1, 2, \dots$. We may assume that, since A is simple and has (SP), we may assume that $r(n, 1) = r(n, 2)$ and

$$[q(n, j, 1)] = [q(n, j', 2)], \quad j, j' = 1, 2, \dots, k(n), \quad n = 1, 2, \dots \quad (\text{e 4.242})$$

To simplify the notation, we may further assume that

$$q(n, j, 1) = q(n, j, 2), \quad j = 1, 2, \dots, k(n), \quad n = 1, 2, \dots \quad (\text{e 4.243})$$

Put

$$\varphi(n, i)'(f) = \sum_{j=k(n)+1}^{r(n, i)} f(\zeta(n, j, i))q(n, j, i) + \varphi(n, i)(f), \quad (\text{e 4.244})$$

$$\varphi(Z, n)(f) = \sum_{j=k(n)+1}^{r(n, 1)} f(\zeta(n, j, 1))q(n, j, 1) + \varphi(Z, n, 1)(f) \text{ and} \quad (\text{e 4.245})$$

$$\varphi(Z, n, 2)'(f) = \sum_{j=k(n)+1}^{r(n, 1)} f(\zeta(n, j, 2))q(n, j, 2) + \varphi(Z, n, 2)(f) \quad (\text{e 4.246})$$

for all $f \in C(\Omega)$. It follows that

$$\limsup_{n \rightarrow \infty} D_c(\varphi(n, 1)', \varphi(Z, n)) \leq \limsup_{n \rightarrow \infty} D_c(\varphi(n, 1), \varphi(Z, n, 1)) = D_c^e(\varphi_X, \varphi_Z). \quad (\text{e 4.247})$$

By (e 4.239), (e 4.243), and the proof of 4.7,

$$\lim_{n \rightarrow \infty} D_c(\varphi(Z, n), \varphi(Z, n, 2)') = 0. \quad (\text{e 4.248})$$

It follows that

$$\limsup_{n \rightarrow \infty} D_c(\varphi(n, 2)', \varphi(Z, n)) \quad (\text{e 4.249})$$

$$\leq \limsup_{n \rightarrow \infty} D_c(\varphi(n, 2)', \varphi(Z, n, 2)') + \lim_{n \rightarrow \infty} D_c(\varphi(Z, n, 2)', \varphi(Z, n)) \quad (\text{e 4.250})$$

$$\leq \limsup_{n \rightarrow \infty} D_c(\varphi(n, 2), \varphi(Z, n, 2)) = D_c^e(\varphi_Y, \varphi_Z). \quad (\text{e 4.251})$$

However, by (e 4.247) and (e 4.249),

$$D_c^e(\varphi_X, \varphi_Y) \leq \limsup_{n \rightarrow \infty} D_c(\varphi(n, 1)', \varphi(n, 2)') \quad (\text{e 4.252})$$

$$\leq \limsup_{n \rightarrow \infty} (D_c(\varphi(n, 1)', \varphi(Z, n)) + D_c(\varphi(n, 2)', \varphi(Z, n))) \quad (\text{e 4.253})$$

$$\leq D_c^e(\varphi_X, \varphi_Z) + D_c^e(\varphi_Z, \varphi_Y). \quad (\text{e 4.254})$$

□

Definition 4.16. Let A be a unital C^* -algebra and let $x, y \in A$ be two normal elements with $\text{sp}(x) = X$ and $\text{sp}(y) = Y$, respectively. Define $\varphi_X, \varphi_Y : C(X \cup Y) \rightarrow A$ to be unital homomorphisms defined by $\varphi_X(f) = f(x)$ and $\varphi_Y(f) = f(y)$ for all $f \in C(X \cup Y)$. We will use the notation $D_c^e(x, y)$ for $D_c^e(\varphi_X, \varphi_Y)$.

5 Approximate unitary equivalence

The purpose of this section is to present 5.6 and 5.5. In the case that A is a unital simple C^* -algebra with $TR(A) = 0$, much more general results were presented in [21]. However, in the spirit of [17], the exact condition for when two normal elements are approximately unitarily equivalent can be obtained in unital simple C^* -algebra A with real rank zero, stable rank one and with weakly unperforated $K_0(A)$. We are also interested in 5.7.

The following is proved in [17].

Theorem 5.1. *Let $\epsilon > 0$. For any unital simple C^* -algebra A of real rank zero with (IR) and any normal element $x \in A$ with $\|x\| \leq 1$ such that*

$$\lambda - x \in \text{Inv}_0(A) \quad (\text{e 5.255})$$

for all $\lambda \in \mathbb{C}$ with $\text{dist}(\lambda, \text{sp}(x)) \geq \epsilon/8$, there is a normal element with finite spectrum $x_0 \in A$ such that

$$\|x - x_0\| < \epsilon. \quad (\text{e 5.256})$$

Proof. This was exactly proved in the proof of the Theorem of [17]. Note that the set

$$F_1 = \{\lambda \in \mathbb{C} : \text{dist}(\lambda, \text{sp}(x)) < r\}$$

in that proof is chosen for $r = \epsilon/8$. □

Lemma 5.2. *Let A be a unital separable simple C^* -algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$, let X be a compact subset of the plane and let $\Delta : (0, 1) \rightarrow (0, 1)$ be an increasing function such that $\lim_{r \rightarrow 0} \Delta(r) = 0$. Then, for any $\epsilon > 0$, there exists $d > 0$ with $d < \epsilon/128$, there exists a finite subset $\{f_1, f_2, \dots, f_n\} \subset C(\overline{X_{d/2}})$ of mutually orthogonal projections with $\sum_{i=1}^n f_i = 1_{C(\overline{X_{d/2}})}$, a finite subset $\mathcal{H} \subset C(\overline{X_{d/2}})_+$ satisfying the following: if $h : C(\overline{X_{d/2}}) \rightarrow A$ is a homomorphism such that*

$$\mu_{\tau \circ h}(O) \geq \Delta(r) \quad (\text{e 5.257})$$

for any open balls O of X with radius $r \geq \epsilon/32$, then for any homomorphism $\varphi : C(\overline{X_{\epsilon/128}}) \rightarrow A$ such that

$$\varphi_{*0}([f_i]) = h_{*0}([f_i]), \quad i = 1, 2, \dots, n, \quad (\text{e 5.258})$$

$$\lambda - h(z), \lambda - \varphi(z) \in \text{Inv}_0(A) \text{ if } \text{dist}(\lambda, X) \geq d, \text{ and} \quad (\text{e 5.259})$$

$$|\tau \circ h(g) - \tau \circ \varphi(g)| < (1/4)\Delta(\epsilon/32) \text{ for all } g \in \mathcal{H}, \quad (\text{e 5.260})$$

then there exists a unitary $u \in A$ such that

$$\|u^* h(z) u - \varphi(z)\| < \epsilon, \quad (\text{e 5.261})$$

where $z \in C(\overline{X_{d/2}})$ is the identity function on $\overline{X_{d/2}}$.

Proof. Let $\epsilon > 0$. We choose $\delta > 0$ which is required by 2.17 for $\epsilon/8$ (with $\overline{X_{\epsilon/16}} = \Omega$). Let

$$d = \min\{\delta/8, \epsilon/2^{21}\}. \quad (\text{e 5.262})$$

Let $\{f'_1, f'_2, \dots, f'_n\} \subset C(\overline{X_{\epsilon/128}})$ be a subset of projections be as required by 4.3 for Δ , $\overline{X_{d/2}}$ (in place of X) and $\epsilon/2$ (instead of ϵ). Define $f_i = f'_i|_{\overline{X_{d/2}}}$, $i = 1, 2, \dots, n$. Now assume A , h and φ be as stated above.. By applying 4.3, one has

$$D_c(h, \varphi) < \epsilon/2. \quad (\text{e 5.263})$$

It follows from 5.1 that, if (e 5.259) holds, there are normal elements x_0 and y_0 with finite spectrum such that

$$\|h(z) - x_0\| < \min\{\epsilon/16, \delta\} \text{ and } \|\varphi(z) - y_0\| < \min\{\epsilon/16, \delta\}. \quad (\text{e 5.264})$$

By 2.17, we have that

$$D_c(h(z), x_0) < \epsilon/8 \text{ and } D_c(\varphi(z), y_0) < \epsilon/8 \quad (\text{e 5.265})$$

Therefore

$$D_c(x_0, y_0) < 3\epsilon/4. \quad (\text{e 5.266})$$

Since A is simple and has real rank zero and stable rank one with weakly unperforated $K_0(A)$, $K_0(A)$ has Riesz interpolation property by a theorem of Zhang [33], by 3.5, there exists a unitary $u \in A$ such that

$$\|u^* x_0 u - y_0\| < 3\epsilon/4. \quad (\text{e 5.267})$$

Combining this with (e 5.264), we conclude that

$$\|u^* h(z) u - \varphi(z)\| < \epsilon. \quad (\text{e 5.268})$$

□

Lemma 5.3. *Let A be a unital infinite dimensional separable simple C^* -algebra with real rank zero, stable rank one and with weakly unperforated $K_0(A)$ and $X \subset \mathbb{C}$ be a compact subset. Let $p_1, p_2, \dots, p_n \in C(X)$ be n mutually orthogonal projections with $\sum_{i=1}^n p_i = 1_{C(X)}$ such that $\{[p_1], [p_2], \dots, [p_n]\}$ generates a subgroup G of $K_0(C(X))$. Suppose that $\kappa_0 : G \rightarrow K_0(A)$ is an order preserving homomorphism with $\kappa_0([1_{C(X)}]) = [1_A]$ and with $\kappa_0([p_i]) > 0$, $i = 1, 2, \dots, n$, and $\kappa_1 : K_1(C(X)) \rightarrow K_1(A)$ is a homomorphism. Then there is a unital monomorphism $\varphi : C(X) \rightarrow A$ such that*

$$\varphi_{*0}|_G = \kappa_0 \text{ and } \varphi_{*1} = \kappa_1. \quad (\text{e 5.269})$$

Proof. Since A is simple and has real rank zero and stable rank one with weakly unperforated $K_0(A)$, as in the proof of 5.2, $K_0(A)$ has Riesz interpolation property and $\rho_A(K_0(A))$ is dense in $\text{Aff}(T(A))$. It follows from [10] that there is a unital simple AH-algebra B with no dimension growth and real rank zero such that

$$(K_0(A), K_0(A)_+, [1_A], K_1(A)) = (K_0(B), K_0(B)_+, [1_B], K_1(B)). \quad (\text{e 5.270})$$

It follows from 4.6 of [19] that there exists a unital embedding $\iota : B \rightarrow A$ which carries the above identification. We will also use the fact that $\rho_B(K_0(B)) = \rho_A(K_0(A))$ is dense in $\text{Aff}(T(A))$. Therefore it suffices to show that the lemma holds for $A = B$. There are mutually orthogonal nonzero projections $e_1, e_2, \dots, e_n \in B$ such that $\kappa_0([p_i]) = e_i$, $i = 1, 2, \dots, n$. Let X_i be the clopen subset of X corresponding to the projection p_i , $i = 1, 2, \dots, n$. Each $e_i B e_i$ is an infinite dimensional unital simple C^* -algebra with $TR(e_i B e_i) = 0$. Therefore there is a monomorphism $\psi_i : C(X_i) \rightarrow e_i B e_i$, $i = 1, 2, \dots, n$. Define a unital monomorphism $\psi : C(X) \rightarrow B$ by $\psi(f) = \sum_{i=1}^n \psi_i(f p_i)$ for all $f \in C(X)$. Note that

$$\kappa_0 = \psi_{*0}|_G. \quad (\text{e 5.271})$$

We also have that ψ_{*0} and $\psi^\sharp : C(X)_{s.a.} \rightarrow \text{Aff}(T(B))$ are compatible and ψ^\sharp is strictly positive. It follows from Corollary 5.3 of [22] that there is a unital monomorphism $\varphi : C(X) \rightarrow B$ such that

$$\varphi_{*0} = \psi_{*0} \text{ and} \quad (\text{e 5.272})$$

$$\varphi_{*1} = \kappa_1. \quad (\text{e 5.273})$$

Lemma follows. □

Lemma 5.4. *Let A be a unital simple C^* -algebra of real rank zero, stable rank one with weakly unperforated $K_0(A)$, let $X \subset \mathbb{C}$ be a compact subset of the plane and let $\Delta : (0, 1) \rightarrow (0, 1)$ be a non-decreasing function such that $\lim_{t \rightarrow 0} \Delta(t) = 0$. For any $1 > r_0 > 0$, any $\epsilon > 0$, any $\eta > 0$, any $\eta_1 > 0$ with $\eta_1 < r_0/4$, any $\eta_2 > 0$ and any finite subset $\mathcal{G} \subset C(\overline{X_{\eta_1}})_+$, where $\overline{X_{\eta_1}} = \{\lambda \in \mathbb{C} : \text{dist}(\lambda, X) < \eta_1\}$, there is a finite subset $\mathcal{H} \subset C(X)_{s.a.}$ satisfying the following: If $x, y \in A$ are normal elements with $\text{sp}(x), \text{sp}(y) \subset X$ such that*

$$|\tau \circ \varphi(g) - \tau \circ \psi(g)| < \eta/2 \text{ for all } g \in \mathcal{H} \text{ and} \quad (\text{e 5.274})$$

$$\mu_{\tau \circ \varphi}(O) \geq \Delta(r) \text{ for all } \tau \in T(A) \quad (\text{e 5.275})$$

for all open balls O of X with radius $r \geq r_0$, where $\varphi, \psi : C(X) \rightarrow A$ are defined by $\varphi(f) = f(x)$ and $\psi(f) = f(y)$ for all $f \in C(X)$, respectively, then there exists $\{\lambda_1, \lambda_2, \dots, \lambda_n\} \subset X$ which is r_0 -dense in X , non-zero mutually orthogonal projections $\{e_1, e_2, \dots, e_n\} \subset A$, two normal elements $x_0, y_0 \in eAe$, where $e = 1 - \sum_{i=1}^n e_i$ and a unitary $u \in A$ such that

$$\|x - (\sum_{i=1}^n \lambda_i e_i + x_0)\| < \epsilon/2, \quad \|u^* y u - (\sum_{i=1}^n \lambda_i e_i + y_0)\| < \epsilon/2 \quad (\text{e 5.276})$$

$$|\tau \circ \varphi_0(g) - \tau \circ \psi_0(g)| < \eta \text{ for all } g \in \mathcal{G} \text{ and for all } \tau \in T(A), \quad (\text{e 5.277})$$

$$\text{sp}(x_0), \text{sp}(y_0) \subset \overline{X_{\eta_1}}, \quad (\text{e 5.278})$$

$$\tau(\sum_{i=1}^n e_i) < \eta_2 \text{ for all } \tau \in T(A) \text{ and} \quad (\text{e 5.279})$$

$$\mu_{\tau \circ \varphi_0}(O) \geq (1/2)\Delta(r/6) \quad (\text{e 5.280})$$

for all open balls $O \subset \overline{X_{\eta_1}}$ with radius $r \geq 3r_0$ and for all $\tau \in T(A)$, where $\varphi_0, \psi_0 : C(\overline{X_{\eta_1}}) \rightarrow A$ is defined by

$$\varphi_0(f) = \sum_{i=1}^n f(\lambda_i) e_i + f(x_0) \text{ and } \psi_0(f) = \sum_{i=1}^n f(\lambda_i) e_i + f(y_0)$$

for all $f \in C(\overline{X_{\eta_1}})$.

Proof. To simplify the notation, we may assume that X is a subset of the unit disk. Note that, since A has real rank zero and stable rank one, so does pAp for any projection $p \in A$. It follows that pAp has (IR) (see [11]). Let $1 > \epsilon > 0$ be given. Let $\epsilon_1 > 0$ be such that $\epsilon/4 > \epsilon_1 > 0$. By [11], there exists $\delta_1 > 0$ such that, for any C^* -algebra D with (IR), any element $z \in D$ with $\|z\| \leq 2$ and

$$\|z^* z - z z^*\| < \delta_1, \quad (\text{e 5.281})$$

then there is a normal element $z_0 \in D$ such that

$$\|z_0 - z\| < \epsilon_1. \quad (\text{e 5.282})$$

Let $\eta, \eta_1, \eta_2 > 0$ be given and a finite subset $\mathcal{G} \subset C(\overline{X_{\eta_1}})_+$ be given. Denote by $\varphi' : C(\overline{X_{\eta_1}}) \rightarrow A$ the homomorphism defined by $\varphi'(f) = f(x)$ for all $f \in C(\overline{X_{\eta_1}})$. Since $\eta_1 < r_0/4$, every open ball of $\overline{X_{\eta_1}}$ of radius $r > r_0$ contains an open balls of X of radius $r/2$. It follows that

$$\mu_{\tau \circ \varphi'}(O) \geq \Delta(r/2) \quad (\text{e 5.283})$$

for all open balls $O \subset \overline{X_{\eta_1}}$ with radius $r > r_0$ and for all $\tau \in T(A)$.

We will applying Lemma 2 of [15]. Note that since A has real rank zero, non-zero projections p_k described in that lemma exists. Thus, we obtain non-zero mutually orthogonal projections $p_1, p_2, \dots, p_n \in A$ and $p'_1, p'_2, \dots, p'_n \in A$ such that

$$\|x - (\sum_{i=1}^n \lambda_i p_i + p x p)\| < \min\{\delta_1/16, \epsilon_1/16\} \text{ and} \quad (\text{e 5.284})$$

$$\|y - (\sum_{i=1}^n \lambda_i p'_i + p' y p')\| < \min\{\delta_1/16, \epsilon_1/16\}, \quad (\text{e 5.285})$$

where $p = \sum_{i=1}^n p_i, p' = \sum_{i=1}^n p'_i$.

Since $\text{sp}(x)$ and $\text{sp}(y)$ are r_0 -dense by (e 5.275), we may assume that $\{\lambda_1, \lambda_2, \dots, \lambda_n\}$ is r_0 -dense. Since A is simple and has real rank zero, there are possibly smaller non-zero projections $e_i \leq p_i$ such that $e_i \lesssim p'_i, i = 1, 2, \dots, n$. In other words, since A has stable rank one, there is a unitary $u \in A$ such that

$$\tau(\sum_{i=1}^n e_i) < \eta_2 \text{ for all } \tau \in T(A), \quad (\text{e 5.286})$$

$$\|x - (\sum_{i=1}^n \lambda_i e_i + \sum_{i=1}^n \lambda_i (p_i - e_i) + p x p)\| < \min\{\delta_1/16, \epsilon_1/16\} \text{ and} \quad (\text{e 5.287})$$

$$\|u^* y u - (\sum_{i=1}^n \lambda_i e_i + y'_0)\| < \min\{\delta_1/16, \epsilon_1/16\}, \quad (\text{e 5.288})$$

where $y'_0 = u^* y_0 u + \sum_{i=1}^n \lambda_i (u^* p'_i u - e_i)$. Put $x'_0 = \sum_{i=1}^n \lambda_i (p_i - e_i) + p x p$. Let $e = 1 - \sum_{i=1}^n e_i$. Then

$$\|(x'_0)^*(x'_0) - (x'_0)(x_0)^*\| < \delta_1 \text{ and } \|(y'_0)^*(y'_0) - (y'_0)(y'_0)^*\| < \delta_1. \quad (\text{e 5.289})$$

By the choice of δ_1 , by applying [11], there exist normal elements $x_0, y_0 \in e A e$ such that

$$\|x_0 - x'_0\| < \epsilon_1 \text{ and } \|y_0 - y'_0\| < \epsilon_1. \quad (\text{e 5.290})$$

It follows that

$$\|x - (\sum_{i=1}^n \lambda_i e_i + x_0)\| < \epsilon_1 \text{ and } \|u^* y u - (\sum_{i=1}^n \lambda_i e_i + y_0)\| < \epsilon_1. \quad (\text{e 5.291})$$

Define $\varphi_0, \psi_0 : C(\overline{X_{\eta_1}}) \rightarrow A$ by

$$\varphi_0(f) = \sum_{i=1}^n f(\lambda_i) e_i + f(x_0) \text{ and } \psi_0(f) = \sum_{i=1}^n f(\lambda_i) e_i + f(y_0)$$

for all $f \in C(\overline{X_{\eta_1}})$. Now, we will choose ϵ_1 sufficiently small to begin with. First, by applying Lemma 3.4 of [24], we will choose \mathcal{H} sufficiently large and σ sufficiently small (independent of A and normal elements given) so that

$$\mu_{\tau \circ \varphi_0}(O) \geq (1/2)\Delta(r/6) \quad (\text{e 5.292})$$

for all open balls O of $\overline{X_{\eta_1}}$ of radius $r \geq 3r_0$, if (e 5.277) holds. In particular, we choose $\mathcal{H} \supset \mathcal{G}$ and $\sigma < \eta/2$. Since \mathcal{G} is finite and given, with sufficiently smaller ϵ_1 , we also have, by (e 5.291), and by assumption (e 5.277),

$$|\tau \circ \varphi_0(g) - \tau \circ \psi_0(g)| < \eta \text{ for all } g \in \mathcal{G} \quad (\text{e 5.293})$$

and for all $\tau \in T(A)$. \square

Theorem 5.5. *Let A be a unital separable simple C^* -algebra of real rank zero, stable rank one with weakly unperforated $K_0(A)$, let $X \subset \mathbb{C}$ be a compact subset of the plane and let $\Delta : (0, 1) \rightarrow (0, 1)$ be a non-decreasing function such that $\lim_{t \rightarrow 0} \Delta(t) = 0$. For any $\epsilon > 0$ there is a finite subset of unitaries $\mathcal{V} \subset C(X)$, a finite subset of projections $\{p_1, p_2, \dots, p_n\} \subset C(X)$, a finite subset $\mathcal{H} \subset C(X)_{s.a.}$, $\sigma > 0$ and $r_0 > 0$ satisfying the following: If $x, y \in A$ are normal elements with $\text{sp}(x), \text{sp}(y) \subset X$ such that*

$$\varphi_{*0}([p_i]) = \psi_{*0}([p_i]) \quad (\text{e 5.294})$$

$$\varphi_{*1}|_{\mathcal{V}} = \psi_{*1}|_{\mathcal{V}}, \quad (\text{e 5.295})$$

$$|\tau \circ \varphi(g) - \tau \circ \psi(g)| < \sigma \text{ for all } g \in \mathcal{H} \text{ and} \quad (\text{e 5.296})$$

$$\mu_{\tau \circ \varphi}(O) \geq \Delta(r) \text{ for all } \tau \in T(A) \quad (\text{e 5.297})$$

for all open balls O of X with radius $r \geq r_0$, where $\varphi, \psi : C(X) \rightarrow A$ are defined by $\varphi(f) = f(x)$ and $\psi(f) = f(y)$ for all $f \in C(X)$, respectively. then there exists a unitary $u \in A$ such that

$$\|u^*xu - y\| < \epsilon. \quad (\text{e 5.298})$$

Proof. To simplify notations, we may assume that X is a compact subset of the unit disk. Let $\Delta_1(r) = (1/64)\Delta(r/12)$ for all $r \in (0, 1)$. Let $\epsilon > 0$. Choose $\mathcal{F} = \{z\}$, where z is the identity function on the unit disk.

Let $\epsilon_0 = \epsilon/64$. Choose $r_0 = \epsilon_0/16$. Let $d > 0$ with $d < \epsilon_0/2^{20}$ be required by 5.2 for ϵ_0 (in place of ϵ) and Δ_1 (in place of Δ). Put $Y = \overline{X_{d/2}}$. Let $\{f_1, f_2, \dots, f_n\} \subset C(Y)$ be a finite subset of mutually orthogonal projections and let $\mathcal{H}_1 \subset C(Y)$ (in place of \mathcal{H}) be required by 5.2 for ϵ_0 (in place of ϵ) and for Δ_1 (in place of Δ). We may assume that $1_{C(Y)} = \sum_{i=1}^n f_i$ and f_i corresponding to $r_0/2^{14}$ -connected components. Note that Y is homeomorphic to a finite CW complex in the plane. Let $\{v_1, v_2, \dots, v_{n_1}\} \subset Y$ be a set of unitaries which generates $K_1(C(Y))$.

Choose $\epsilon_1 > 0$ satisfying the following: if x', y' be two normal elements in a unital C^* -algebra B with $\text{sp}(x'), \text{sp}(y') \subset Y$ and

$$\|x' - y'\| < \epsilon_1,$$

then

$$(\varphi')_{*0}([f_i]) = (\psi')_{*0}([f_i]), \quad i = 1, 2, \dots, n \text{ and} \quad (\text{e 5.299})$$

$$(\varphi')_{*1} = (\psi')_{*1}, \quad (\text{e 5.300})$$

where $\varphi', \psi' : C(Y) \rightarrow B$ are defined by $\varphi'(f) = f(x')$ and $\psi'(f) = f(y')$ for all $f \in C(Y)$. Let $\epsilon_2 = \min\{\epsilon_1/4, \epsilon_0/2\}$. Let $\eta = (1/2^{10})(\Delta_1(\epsilon_0/64))$, $\eta_1 = \min\{d/2, r_0/64\}$ and $\eta_2 = \eta$.

Let $\mathcal{H} \subset C(X)_{s.a.}$ be a finite subset be required by 5.4 for r_0, ϵ_2 (in place of ϵ), $\eta, \eta_1, \eta_2, \mathcal{H}_1$ (in place of \mathcal{G}) and Δ .

Let $p_i = f_i|_X$, $i = 1, 2, \dots, n$ and let $u_j = v_j|_X$, $j = 1, 2, \dots, n_1$. Put $\mathcal{V} = \{u_1, u_2, \dots, u_{n_1}\}$. Let $\sigma = \eta/2$. Now suppose that x, y are two normal elements in A satisfying the assumption for the above $\mathcal{V}, \{p_1, p_2, \dots, p_n\}, \mathcal{H}, \sigma$ and r_0 .

By 5.4, there exists $\{\lambda_1, \lambda_2, \dots, \lambda_m\} \subset X$ which is r_0 -dense, there are non-zero mutually orthogonal projections $\{e_1, e_2, \dots, e_m\} \subset A$, two normal elements $x_0, y_0 \in eAe$, where $e = 1 -$

$\sum_{i=1}^m e_i$ and a unitary $w \in A$ such that

$$\|x - (\sum_{i=1}^m \lambda_i e_i + x_0)\| < \epsilon_2/2, \quad \|w^* y w - (\sum_{i=1}^m \lambda_i e_i + y_0)\| < \epsilon_2/2, \quad (\text{e 5.301})$$

$$|\tau \circ \varphi_0(g) - \tau \circ \psi_0(g)| < \eta \text{ for all } g \in \mathcal{H}_1 \text{ and for all } \tau \in T(A), \quad (\text{e 5.302})$$

$$\text{sp}(x_0), \text{sp}(y_0) \subset \overline{X_{\eta_1}}, \quad (\text{e 5.303})$$

$$\tau(\sum_{i=1}^m e_i) < \eta_2 \text{ for all } \tau \in T(A) \text{ and} \quad (\text{e 5.304})$$

$$\mu_{\tau \circ \varphi_0}(O) \geq (1/2)\Delta(r/6) \quad (\text{e 5.305})$$

for all open balls $O \subset \overline{X_{\eta_1}}$ with radius $r \geq 3r_0$ and for all $\tau \in T(A)$, where $\varphi_0, \psi_0 : C(\overline{X_{\eta_1}}) \rightarrow A$ is defined by

$$\varphi_0(f) = \sum_{i=1}^m f(\lambda_i) e_i + f(x_0) \text{ and } \psi_0(f) = \sum_{i=1}^m f(\lambda_i) e_i + f(y_0)$$

for all $f \in C(\overline{X_{\eta_1}})$.

Since A is a unital simple C^* -algebra with real rank zero, there are, for each i , non-zero mutually orthogonal projections $e_{i,0}, e_{i,1}, e_{i,2}$ such that

$$e_i = e_{i,0} + e_{i,1} + e_{i,2} \text{ and } 9\tau(\sum_{i=1}^n (e_{i,1} + e_{i,2})) < \tau(e_j) \quad (\text{e 5.306})$$

for all $\tau \in T(A)$ and $j = 1, 2, \dots, m$. Define

$$\varphi'_0(f) = \sum_{i=1}^m f(\lambda_i) e_i, \quad \varphi_{0,0}(f) = \sum_{i=1}^m f(\lambda_i) e_{i,0}, \quad (\text{e 5.307})$$

$$\varphi_{0,1}(f) = \sum_{i=1}^m f(\lambda_i) e_{i,1} \text{ and } \varphi_{0,2} = \sum_{i=1}^m f(\lambda_i) e_{i,2} \quad (\text{e 5.308})$$

for all $f \in C(Y)$.

Put $P_1 = \sum_{i=1}^m e_{i,1}$ and $P_2 = \sum_{i=1}^n e_{i,2}$. We have

$$\tau(P_1 + P_2) < \eta_2/8 \text{ for all } \tau \in T(A). \quad (\text{e 5.309})$$

It follows from 5.3 that there are unital monomorphisms $H_1 : C(Y) \rightarrow P_1 A P_1$ and $H_2 : C(Y) \rightarrow P_2 A P_2$ such that

$$(H_1)_*([f_i]) = (\varphi_{0,1})_*([f_i]), \quad (H_2)_*([f_i]) = (\varphi_{0,2})_*([f_i]), \quad i = 1, 2, \dots, n \text{ and} \quad (\text{e 5.310})$$

$$(H_1)_*([v_j]) = -(\varphi_x)_*([v_j]) \text{ and } (H_2)_*([v_j]) = (\varphi_x)_*([v_j]), \quad j = 1, 2, \dots, n_1, \quad (\text{e 5.311})$$

where $\varphi_x : C(Y) \rightarrow A$ is defined by $\varphi_x(f) = f(x)$ for all $f \in C(Y)$. Let $x_1 = H_1(z) + H_2(z)$, where z is the identity function on Y . Then, by 5.1, there are $\{\mu_1, \mu_2, \dots, \mu_{m_1}\}$ which is $r_0/2^{12}$ -dense and mutually orthogonal non-zero projections $\{e_{1,3}, e_{2,3}, \dots, e_{m_1,3}\}$ in $(P_1 + P_2)A(P_1 + P_2)$ such that

$$\|x_1 - \sum_{j=1}^{m_1} \mu_j e_{j,3}\| < r_0/2^{12}. \quad (\text{e 5.312})$$

By 4.4, there is a unitary $v \in (1 - e)A(1 - e)$ such that

$$\|v^* \varphi'_0(z)v - (x_1 + \varphi_{0,0}(z))\| < r_0/2^{11}. \quad (\text{e 5.313})$$

Define $\varphi'_x, \psi'_y : C(Y) \rightarrow (1 - P_2)A(1 - P_2)$ by

$$\varphi'_x(f) = H_1(f) + f(x_0) + \varphi_{00}(f) \text{ and } \psi'_y(f) = H_1(f) + f(y_0) + \varphi_{00}(f) \quad (\text{e 5.314})$$

for all $f \in C(Y)$. We have, for all $\lambda \in \mathbb{C}$,

$$\lambda - \varphi'_x(z), \lambda - \psi'_y(z) \in \text{Inv}_0((1 - P_2)A(1 - P_2)) \quad (\text{e 5.315})$$

if $\lambda \notin \overline{X_{d/2}}$. By the choice of ϵ_1 , (e 5.301) and (e 5.310), and the fact that A has stable rank one, we check that

$$(\varphi'_x)_*0([f_i]) = (\psi'_y)_*0([f_i]), \quad i = 1, 2, \dots, n. \quad (\text{e 5.316})$$

For each open ball $O \subset \overline{X_{d/2}}$ with radius $r > r_0$, we estimate that, by (e 5.304) and (e 5.305),

$$\mu_{\tau \circ \varphi'_x}(O) > \mu_{\tau \circ \varphi_0}(O) - \tau(P_1) - \tau(P_2) \quad (\text{e 5.317})$$

$$\geq (1/2)\Delta(r/6) - \eta_2 \quad (\text{e 5.318})$$

$$\geq (1/2)\Delta(r/6) - 2^{-10}\Delta_1(\epsilon_0/64) \quad (\text{e 5.319})$$

$$> 1/4\Delta(r/6) \geq \Delta_1(r) \quad (\text{e 5.320})$$

for all $r \geq 3r_0$ and for all $\tau \in T(A)$. It follows that

$$\mu_{t \circ \varphi'_x}(O) > \Delta_1(r) \text{ for all } t \in T((1 - P_2)A(1 - P_2)) \quad (\text{e 5.321})$$

and for all open balls O with radius $r > 3r_0$. For $f \in \mathcal{H}_1$, by (e 5.304) and (e 5.302)

$$|\tau \circ \varphi'_x(f) - \tau \circ \psi'_y(f)| < |\tau \circ \varphi_0(f) - \tau \circ \psi_0(f)| + 2\tau(1 - e) \quad (\text{e 5.322})$$

$$< \eta + 2\eta_2 \leq (3/2^{10})(\Delta_1(\epsilon_0/64)) \quad (\text{e 5.323})$$

for all $\tau \in T(A)$. It follows that

$$|t \circ \varphi'_x(f) - t \circ \psi'_y(f)| < \frac{(3/2^{10})(\Delta_1(\epsilon_0/64))}{1 - \eta} < (1/4)\Delta(\epsilon_0/64) \quad (\text{e 5.324})$$

for all $t \in T((1 - P_2)A(1 - P_2))$.

It follows from 5.2 that there is a unitary $w_0 \in (1 - P_2)A(1 - P_2)$ such that

$$\|w_0^* \varphi'_y(z)w_0 - \varphi'_x(z)\| < \epsilon_0. \quad (\text{e 5.325})$$

Put $w_1 = v + e$ and $w_2 = w_0 + (1 - P_2)$ and $u = w_1 w_2^* w_1^*$. Then, by (e 5.301), (e 5.313), (e 5.307) and (e 5.313) again,

$$x \approx_{\epsilon_2/2} \varphi_0(z) \approx_{r_0/2^{11}} w_1(x_1 + \varphi_{00}(z) + x_0)w_1^* \quad (\text{e 5.326})$$

$$= w_1(H_2(z) + H_1(z) + \varphi_{00}(z) + x_0)w_1^* \quad (\text{e 5.327})$$

$$\approx_{\epsilon_0} w_1(H_2(z) + w_2^* \varphi'_y(z)w_2)w_1^* \quad (\text{e 5.328})$$

$$= w_1 w_2^*(H_2(z) + H_1(z) + \varphi_{00}(z) + y_0)w_2 w_1^* \quad (\text{e 5.329})$$

$$\approx_{r_0/2^{11}} w_1 w_2^*(w_1^*(\varphi'_0(z) + y_0)w_1)w_2 w_1^* \quad (\text{e 5.330})$$

$$\approx_{\epsilon_2/2} u y u^*. \quad (\text{e 5.331})$$

But $\epsilon_2/2 + r_0/2^{11} + \epsilon_0 + r_0/2^{11} + \epsilon_2/2 < \epsilon$. □

Theorem 5.6. *Let A be a unital separable simple C^* -algebra of real rank zero, stable rank one and weakly unperforated $K_0(A)$ and let $x \in A$ be a normal element with $\text{sp}(x) = X$. For any $\epsilon > 0$ there is a finite subset $\mathcal{V} \subset K_1(C(\text{sp}(x)))$, a finite subset $\mathcal{P} \subset K_0(C(\text{sp}(x)))$, a finite subset $\mathcal{H} \subset C(\text{sp}(x))_{s.a.}$, $\sigma > 0$ satisfying the following: If $y \in A$ is normal element with $\text{sp}(y) \subset X$ such that*

$$\varphi_{*0}|_{\mathcal{P}} = \psi_{*0}|_{\mathcal{P}} \quad (\text{e } 5.332)$$

$$\varphi_{*1}|_{\mathcal{V}} = \psi_{*1}|_{\mathcal{V}} \text{ and} \quad (\text{e } 5.333)$$

$$|\tau \circ \varphi(g) - \tau \circ \psi(g)| < \sigma \text{ for all } g \in \mathcal{H} \text{ for all } \tau \in T(A), \quad (\text{e } 5.334)$$

then there exists a unitary $u \in A$ such that

$$\|u^*xu - y\| < \epsilon. \quad (\text{e } 5.335)$$

Theorem 5.7. *Let A be a unital separable simple C^* -algebra with real rank zero, stable rank one and with weakly unperforated $K_0(A)$. Let $x, y \in A$ be two normal elements. Suppose that $D_c(x, y) = 0$ and $[\lambda - x] = [\lambda - y]$ in $K_1(A)$ for all $\lambda \notin X \cup Y$. Then*

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) = 0 \quad (\text{e } 5.336)$$

Proof. Since A is a unital simple C^* -algebra, the assumption of $D_c(x, y) = 0$ implies that $\text{sp}(x) = \text{sp}(y) = X$. Let $\varphi, \psi : C(X) \rightarrow A$ be the unital monomorphisms induced by x and y , respectively. The assumption implies that $\varphi_{*1} = \psi_{*1}$. It follows from 2.16 that we also have $\varphi_{*0} = \psi_{*0}$. Thus the theorem follows from 5.6. \square

6 Distance between unitary orbits for normal elements with non-zero K_1

Let A be a unital separable simple C^* -algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$. 5.7 provides a clue how to described an upper bound for the distance between unitary orbits for normal elements in A . If two normal elements $x, y \in A$ have the same spectrum and induce the same homomorphism from $K_1(C(\text{sp}(x)))$ to $K_1(A)$, then an upper bound for the distance between their unitary orbits can be similarly described. When they have different spectrum and with non-trivial K_1 information, however, things are very different. This section deals with the case that $(\lambda - x)^{-1}(\lambda - y) \in \text{Inv}_0(A)$ for all $\lambda \notin \text{sp}(x) \cup \text{sp}(y)$. Note that the assumption allows the case that $\lambda - x \notin \text{Inv}_0(A)$ for $\lambda \in Y \setminus X$ and $\lambda - y \notin \text{Inv}_0(A)$ for $\lambda \in X \setminus Y$. This can be done partly because we are able to borrow a Mayer-Vietoris Theorem.

Definition 6.1. Let A be a unital simple C^* -algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$ and let Ω be a compact metric space. Let F_1 and F_2 be two finite subsets of Ω . Suppose that $\kappa_{F_1}, \kappa_{F_2} \in H_{c,1}(C(\Omega), A)_+$ are two elements represented by two homomorphisms whose spectra are F_1 and F_2 , respectively. Suppose also that $\kappa_{F_1}(f_O)$ and $\kappa_{F_2}(f_O)$ are projections for all open subsets $O \subset \Omega$.

Suppose that $F_1 = \{x_1, x_2, \dots, x_m\}$ and $F_2 = \{y_1, y_2, \dots, y_n\}$. Suppose that

$$D_c(\kappa_{F_1}, \kappa_{F_2}) = r. \quad (\text{e } 6.337)$$

Then, as proved earlier in 3.4, for any $\epsilon > 0$, there are $a_{i,j} \in W(A)$, $1 \leq i \leq m$ and $1 \leq j \leq n$ such that

$$\sum_{j=1}^n a_{i,j} = \kappa_{F_1}([f_{\{x_i\}}]), \quad \sum_{i=1}^m a_{i,j} = \kappa_{F_2}([f_{\{y_j\}}]) \text{ and} \quad (\text{e } 6.338)$$

$$|x_i - y_j| \leq r + \epsilon, \text{ if } a_{i,j} \neq 0 \quad (\text{e } 6.339)$$

By a paring of κ_{F_1} and κ_{F_2} we mean a subset $R(\kappa_{F_1}, \kappa_{F_2}) \subset \{1, 2, \dots, m\} \times \{1, 2, \dots, n\}$ of those pairs of (i, j) such that (e 6.338) and (e 6.339) hold.

Definition 6.2. Given a pair of κ_1 and κ_2 with spectra X and Y , we say that the pair has a *hub* at $X \cap Y$, if $X = \sqcup_{i=1}^{m_1} S_i$ and $Y = \sqcup_{k=1}^{m_2} G_k$, where $\{S_1, S_2, \dots, S_{m_1}\}$ is a set of mutually disjoint clopen subsets of X and $\{G_1, G_2, \dots, G_{m_2}\}$ is a set of mutually disjoint clopen subsets of Y , there exists $\epsilon_0 > 0$ such that, for any $0 < \epsilon < \epsilon_0$, there are finite ϵ -approximations κ_{F_1} of κ_1 and κ_{F_2} of κ_2 satisfying the following: There is a pairing $R(\kappa_{F_1}, \kappa_{F_2})$ such that, for each pair (t, k) with $S_t \cap G_k \neq \emptyset$, there is a pair $(i, j) \in R(\kappa_{F_1}, \kappa_{F_2})$ such that $x_i, y_j \in S_t \cap G_k$.

Obvious examples that any pairs (κ_1, κ_2) have a hub at $X \cap Y$ are when $X \cap Y = \emptyset$, or when $X = Y$ are connected. More examples will be presented in 6.10.

Let $x, y \in A$ be two normal elements with $X = \text{sp}(x)$ and $Y = \text{sp}(y)$. Denote by $\varphi_X : C(X) \rightarrow A$ and $\varphi_Y : C(Y) \rightarrow A$ the homomorphisms induced by x and y . We say the *pair* (x, y) has a *hub* at $X \cap Y$, if the pair (φ_X, φ_Y) has a hub at $X \cap Y$.

Lemma 6.3. *Let A be a unital separable simple C^* -algebra with real rank zero, stable rank one and with weakly unperforated $K_0(A)$, and let $x, y \in A$ be two normal elements with $X = \text{sp}(x)$ and $Y = \text{sp}(y)$. For any $\epsilon > 0$, any finite subset $\mathcal{G}_X \subset C(X)_{s.a.}$ and any finite subset $\mathcal{G}_Y \subset C(Y)_{s.a.}$, there exist mutually orthogonal projections $\{e_1, e_2, \dots, e_n\} \subset A$ with $\sum_{i=1}^n e_i = 1_A$, $\lambda_1, \lambda_2, \dots, \lambda_n \in \text{sp}(x)$ and $\mu_1, \mu_2, \dots, \mu_n \in \text{sp}(y)$ such that*

$$\max\{|\tau \circ g(x) - \tau \circ g(x_1)| : g \in \mathcal{G}_X\} < \epsilon/2, \quad (\text{e 6.340})$$

$$\max\{|\tau \circ g(y) - \tau \circ g(y_1)| : g \in \mathcal{G}_Y\} < \epsilon/2 \text{ for all } \tau \in T(A), \quad (\text{e 6.341})$$

$$D_c(x, x_1) \leq D_c^e(x, x_1) < \epsilon/2, \quad D_c(y, y_1) \leq D_c^e(y, y_1) < \epsilon/2, \quad (\text{e 6.342})$$

$$D_c^e(x_1, y_1) < D_c^e(x, y) + \epsilon \text{ and } \|x_1 - y_1\| < D_c(x, y) + \epsilon, \quad (\text{e 6.343})$$

where

$$x_1 = \sum_{i=1}^n \lambda_i e_i, \quad y_1 = \sum_{i=1}^n \mu_i e_i \quad (\text{e 6.344})$$

and

$$\max_{1 \leq i \leq n} |\lambda_i - \mu_i| \leq D_c(x, y) + \epsilon \quad (\text{e 6.345})$$

Moreover, if $X \cap Y \neq \emptyset$, for any $\sigma > 0$ and $\eta > 0$, we may require that

$$x_1 = \sum_{i=1}^{m_0} \lambda_i e(i, 0) + x_{1,1} \text{ and } y_1 = \sum_{i=1}^{m_0} \lambda_i e(i, 0) + y_{1,1}, \quad (\text{e 6.346})$$

$$\sum_{i=1}^{m_0} \tau(e(i, 0)) < \sigma \text{ for all } \tau \in T(A), D_c(x_{1,1}, y_{1,1}) \leq D_c^e(x, y) + \epsilon \quad (\text{e 6.347})$$

where $\{e(1, 0), e(2, 0), \dots, e(m_0, 0)\}$ is a set of mutually orthogonal projections, $\{\lambda_1, \lambda_2, \dots, \lambda_{m_0}\}$ is η -dense in $X \cap Y$, $x_{1,1}, y_{1,1} \in (1 - \sum_{i=1}^{m_0} e(i, 0))A(1 - \sum_{i=1}^{m_0} e(i, 0))$ are normal elements with finite spectrum in X and Y , respectively,

In the above, if $X = \sqcup_{i=1}^{m_1} F_j$ and $Y = \sqcup_{k=1}^{m_2} G_k$, where F_1, F_2, \dots, F_{m_1} are $\eta/2$ -connected components of X and G_1, G_2, \dots, G_{m_2} are $\eta/2$ -connected components of Y , we may assume that $\{\lambda_i\}$ is η -dense in X and $\{\mu_i\}$ is η -dense in Y . In particular, we may require that $\{\lambda_i\} \cap F_j \neq \emptyset$, $\{\mu_i\} \cap G_k \neq \emptyset$. Moreover, if $F_j \cap G_k \neq \emptyset$, we may further assume that there exist $\lambda_{i(j)}, \mu_{i(k)} \in F_j \cap G_k$ such that $|\lambda_{i(j)} - \mu_{i(k)}| < \eta$.

Furthermore, if the pair (x, y) has a hub at $X \cap Y$, then, we may require that, for each pair (j, k) with $F_j \cap G_k \neq \emptyset$, there are $\lambda_i, \mu_i \in F_j \cap G_k$.

Proof. The main part of this lemma follows from 4.13. In fact that the existence of x_1 and y_2 satisfy everything up to (e 6.342) follows immediately from 4.13. By the assumption,

$$D_c(x_1, y_1) \leq D_c(x_1, x) + D_c(x, y) + D_c(y, y_1) < D_c(x, y) + \epsilon.$$

Note that $\text{sp}(x_1) \cap Y, Y \cap X \subset X$. It follows from 4.15 that

$$D_c^e(x_1, y) \leq D_c^e(x_1, x) + D_c^e(x, y). \quad (\text{e 6.348})$$

Similarly

$$D_c^e(x_1, y_1) \leq D_c^e(x_1, y) + D_c^e(y, y_1). \quad (\text{e 6.349})$$

Therefore

$$D_c^e(x_1, y_1) \leq D_c^e(x, y) + D_c^e(x_1, x) + D_c^e(y, y_1). \quad (\text{e 6.350})$$

Applying 3.5, there exists a unitary u , such that $\|x_1 - u^*y_1u\| < D_c(x, y) + \epsilon$. Let y_1 be replaced by u^*y_1u , we may assume (e 6.343) holds.

Further by applying the proof of 3.5, (e 6.344) and (e 6.345) hold. The second part of the statement with (e 6.346) and (e 6.347) follows from the definition of $D_c^e(\cdot, \cdot)$ and 4.11.

The third and fourth parts of the statement follow from (e 6.346) and the fact that the finite η -approximations of ψ_X and ψ_Y can be made for arbitrarily small η .

Suppose, in addition, that the pair (x, y) has a hub at $X \cap Y$. For each pair (j, k) with $F_j \cap G_k \neq \emptyset$, we may assume that, by choosing sufficiently better finite approximation, without loss of generality, that, there is $\lambda_{i'} \in F_j \cap G_k$ and there is $\mu_{i''} \in F_j \cap G_k$. By the assumption that the pair (x, y) has a hub at $X \cap Y$ and its definition, we may further assume, there are $\lambda_i, \mu_i \in F_j \cap G_k$. \square

Corollary 6.4. *Let A be a unital separable simple C^* -algebra of real rank zero, stable rank one and weakly unperforated $K_0(A)$ and let $x, y \in A$ be normal elements. Then*

$$D_c(x, y) \leq D_c^e(x, y) \leq 2D_c(x, y). \quad (\text{e 6.351})$$

Proof. We will prove the second inequality.

Put $X = \text{sp}(x)$ and $Y = \text{sp}(y)$. By 6.3, there are two sequences of normal elements $x_k, y_k \in A$ with

$$\lim_{k \rightarrow \infty} D_c^e(x_k, x) = 0, \quad \lim_{k \rightarrow \infty} D_c^e(y_k, y) = 0, \quad (\text{e 6.352})$$

$$x_k = \sum_{i=1}^{m(k)} \lambda(k, i) p(k, i) \quad \text{and} \quad y_k = \sum_{i=1}^{m(k)} \mu(k, i) p(k, i), \quad (\text{e 6.353})$$

where $\lambda(k, i) \in X$ and $\mu(k, i) \in Y$,

$$\lim_{k \rightarrow \infty} \max_{\{1 \leq i \leq m(k)\}} |\lambda(k, i) - \mu(k, i)| \leq D_c(x, y) \quad (\text{e 6.354})$$

and $\{p(k, 1), p(k, 2), \dots, p(k, m(k))\}$ is a sequence of mutually orthogonal non-zero projections in A with $\sum_{i=1}^{m(k)} p(k, i) = 1_A$, $k = 1, 2, \dots$. Note that

$$\lim_{k \rightarrow \infty} D_c(x_k, y_k) = D_c(x, y). \quad (\text{e 6.355})$$

We may write

$$x_k = \sum_{i=1}^{m(k)} \lambda(k, i) p'(k, i) + \sum_{i=1}^{m(k)} \lambda(k, i) p''(k, i) \quad \text{and} \quad (\text{e 6.356})$$

$$y_k = \sum_{i=1}^{m(k)} \mu(k, i) p'(k, i) + \sum_{i=1}^{m(k)} \mu(k, i) p''(k, i), \quad (\text{e 6.357})$$

where $p'(k, i), p''(k, i) \neq 0$ and $p'(k, i) + p''(k, i) = p(k, i)$, $i = 1, 2, \dots, m(k)$, $k = 1, 2, \dots$. To simplify the notation, we may write that

$$x_k = \sum_{i=1}^{2m(k)} \lambda(k, i) p(k, i) \quad \text{and} \quad y_k = \sum_{i=1}^{2m(k)} \mu(k, i) p(k, i), \quad (\text{e 6.358})$$

$$\lambda(k, m(k) + i) = \lambda(k, i) \quad \text{and} \quad \mu(k, m(k) + i) = \mu(k, i), \quad i = 1, 2, \dots, m(k), \quad (\text{e 6.359})$$

where $p(k, i) \neq 0$ for all i and k . Without loss of generality, we may assume that $S_k = \{\lambda(k, 1), \lambda(k, 2), \dots, \lambda(k, r(k))\} \subset X \cap Y$ and $\lambda(k, j) \notin Y$, $r(k) < j \leq 2m(k)$. Let $T_k = \{\mu(k, \alpha(1)), \mu(k, \alpha(2)), \dots, \mu(k, \alpha(g(k)))\} \subset X \cap Y$ and $\mu(k, j) \notin X$ if $j \neq \alpha(i)$ ($1 \leq i \leq g(k)$). We may also assume that S_k and T_k both are ϵ_k -dense in $X \cap Y$ and $\lim_{k \rightarrow \infty} \epsilon_k = 0$. A standard argument allows us to assume, without loss of generality, that $\lambda(k, i) = \mu(k, \alpha(i))$, $i = 1, 2, \dots, f(k)$, where $f(k) \leq \min\{r(k), g(k)\}$ and $W_k = \{\lambda(k, i) : 1 \leq i \leq f(k)\}$ is δ_k -dense in $X \cap Y$ and $\lim_{k \rightarrow \infty} \delta_k = 0$. By (e 6.358) and (e 6.359), we may assume that $\alpha(i) > f(k)$, $i = 1, 2, \dots, f(k)$. In particular,

$$\{\alpha(i) : 1 \leq i \leq f(k)\} \cap \{1, 2, \dots, f(k)\} = \emptyset. \quad (\text{e 6.360})$$

Since A is simple and has (SP), there is a sequence of finite sets of non-zero projections $e(k, i) \leq p(k, i)$ such that

$$p(k, i) - e(k, i) \neq 0, [e(k, i)] = [e(k, 1)], \quad i = 1, 2, \dots, f(k), \quad (\text{e 6.361})$$

$$\text{and} \quad \sum_{i=1}^{m(k)} \tau(e(k, i)) < 1/k \quad \text{for all } \tau \in T(A), \quad (\text{e 6.362})$$

$k = 1, 2, \dots$. Let $u_k \in U(A)$ be a sequence of unitaries such that

$$u_k^* e(k, i) u_k = e(k, \alpha(i)), \quad u_k e(k, \alpha(i)) u_k^* = e(k, i), \quad i = 1, 2, \dots, f(k), \quad (\text{e 6.363})$$

$$u_k^* (p(k, i) - e(k, i)) u_k = (p(k, i) - e(k, i)), \quad (\text{e 6.364})$$

$$u_k (p(k, \alpha(i)) - e(k, \alpha(i))) u_k^* = (p(k, \alpha(i)) - e(k, \alpha(i))), \quad (\text{e 6.365})$$

$$i = 1, 2, \dots, f(k) \quad \text{and} \quad (\text{e 6.366})$$

$$u_k^* p(k, j) u_k = p(k, j), \quad \text{if } j \notin \{i : 1 \leq i \leq f(k)\}. \quad (\text{e 6.367})$$

Define

$$x_{0,k} = \sum_{i=1}^{f(k)} \lambda(k, i) e(k, i), \quad (\text{e 6.368})$$

$$x_{1,k} = \sum_{i=1}^{f(k)} \lambda(k, i) (p(k, i) - e(k, i)) + \sum_{i=f(k)+1}^{2m(k)} \lambda(k, i) p_{k,i} \quad (\text{e 6.369})$$

$$y_{0,k} = \sum_{i=1}^{f(k)} \mu(k, \alpha(i)) e(k, \alpha(i)) \quad (\text{e 6.370})$$

$$y_{1,k} = \sum_{i=1}^{f(k)} \mu(k, i) (p(k, \alpha(i)) - e(k, \alpha(i))) + \sum_{i=f(k)+1}^{2m(k)} \mu(k, i) p(k, i). \quad (\text{e 6.371})$$

Note that

$$u_k y_{0,k} u_k^* = \sum_{i=1}^{f(k)} \mu(k, \alpha(i)) u_k e(k, \alpha(i)) u_k^* = x_{0,k}.$$

Note now that $\lambda(k, i) = \mu(k, \alpha(i))$, $i = 1, 2, \dots, f(k)$, W_k is δ_k -dense in $X \cap Y$ and

$$\lim_{k \rightarrow \infty} \sup \left\{ \tau \left(\sum_{i=1}^{f(k)} e(k, i) \right) : \tau \in T(A) \right\} = 0 \quad (\text{e 6.372})$$

Moreover we still have that

$$\lim_{k \rightarrow \infty} D_c(x, x_{0,k} + x_{1,k}) = 0 \text{ and } \lim_{k \rightarrow \infty} D_c(y, y_{0,k} + y_{1,k}) = 0. \quad (\text{e 6.373})$$

Therefore

$$D_c^e(x, y) \leq \liminf_{k \rightarrow \infty} D_c(x_{1,k}, y_{1,k}). \quad (\text{e 6.374})$$

Note also, by (e 6.360)

$$x_{1,k} = \sum_{i=1}^{f(k)} \lambda(k, i) (p(k, i) - e(k, i)) + \sum_{i=1}^{f(k)} \lambda(k, \alpha(i)) (p(k, \alpha(i)) - e(k, \alpha(i))) \quad (\text{e 6.375})$$

$$+ \sum_{i=1}^{f(k)} \lambda(k, \alpha(i)) e(k, \alpha(i)) + \sum_{j \notin \{i, \alpha(i) : 1 \leq i \leq f(k)\}} \lambda(k, j) p(k, j) \text{ and} \quad (\text{e 6.376})$$

$$y_{1,k} = \sum_{i=1}^{f(k)} \mu(k, \alpha(i)) (p(k, \alpha(i)) - e(k, \alpha(i))) + \sum_{j \notin \{\alpha(i) : 1 \leq i \leq f(k)\}} \mu(k, j) p(k, j) \quad (\text{e 6.377})$$

$$= \sum_{i=1}^{f(k)} \mu(k, i) (p(k, i) - e(k, i)) + \sum_{i=1}^{f(k)} \mu(k, \alpha(i)) (p(k, \alpha(i)) - e(k, \alpha(i))) \quad (\text{e 6.378})$$

$$+ \sum_{i=1}^{f(k)} \mu(k, i) e(k, i) + \sum_{j \notin \{i, \alpha(i) : 1 \leq i \leq f(k)\}} \mu(k, j) p(k, j). \quad (\text{e 6.379})$$

Since, for $i = 1, 2, \dots, f(k)$, $[e_{k,i}] = [e_{k, \alpha(i)}]$, $\lambda(k, i) = \mu(k, \alpha(i))$ and

$$\limsup_{k \rightarrow \infty} |\lambda(k, \alpha(i)) - \mu(k, i)| \quad (\text{e 6.380})$$

$$\leq \limsup_{k \rightarrow \infty} (|\lambda(k, \alpha(i)) - \mu(k, \alpha(i))| + |\mu(k, \alpha(i)) - \mu(k, i)|) \quad (\text{e 6.381})$$

$$= \limsup_{k \rightarrow \infty} |\lambda(k, \alpha(i)) - \mu(k, \alpha(i))| + \limsup_{k \rightarrow \infty} |\lambda(k, i) - \mu(k, i)| \quad (\text{e 6.382})$$

$$\leq D_c(x, y) + D_c(x, y). \quad (\text{e 6.383})$$

It follows that (by comparing the last expresses of $x_{1,k}$ and $y_{1,k}$ above and using (e 6.354))

$$D_c(x_{1,k}, y_{1,k}) \leq 2D_c(x, y). \quad (\text{e 6.384})$$

Therefore, by (e 6.374),

$$D_c^e(x, y) \leq 2D_c(x, y). \quad (\text{e 6.385})$$

□

The following is a useful observation for the proof of the main results in this section.

Lemma 6.5. *Let A be a unital C^* -algebra and let X and Y be two compact subsets of the plane. Suppose that $x, y \in A$ are two normal elements with $\text{sp}(x) = X$ and $\text{sp}(y) = Y$ and suppose that $\varphi_X : C(X) \rightarrow A$ and $\varphi_Y : C(Y) \rightarrow A$ are induced unital monomorphisms by x and by y , respectively. Suppose also that $[\lambda - x] = [\lambda - y]$ in $K_1(A)$ for all $\lambda \notin X \cup Y$. Then*

$$(\varphi_X \circ \iota_1)_{*1} = 0, \quad (\text{e 6.386})$$

where $I = \{f \in C(X) : f|_{X \cap Y} = 0\}$ and $\iota_1 : I \rightarrow C(X)$ is the embedding.

Consequently, if $X \cap Y = \emptyset$, then $I = C(X)$ and $(\varphi_X)_{*1} = 0$.

In this case (e 6.386) means that

Proof. Let $\pi_X : C(X \cup Y) \rightarrow C(X)$ and $\pi_Y : C(X \cup Y) \rightarrow C(Y)$ be quotient maps. Define $\psi_1 = \varphi_X \circ \pi_X$ and $\psi_2 = \varphi_Y \circ \pi_Y$. The assumption implies that

$$(\psi_1)_{*1} = (\psi_2)_{*1}. \quad (\text{e 6.387})$$

Put

$$J = \{f \in C(X \cup Y) : f|_Y = 0\}.$$

Note that $J \cong C_0(X \cup Y \setminus Y)$. But $X \cup Y \setminus Y = X \setminus X \cap Y$. Therefore there is a natural isomorphism $h : I \rightarrow J$. Let $\iota_2 : J \rightarrow C(X \cup Y)$ be the embedding. Then

$$\pi_X \circ \iota_2 \circ h = \iota_1. \quad (\text{e 6.388})$$

Thus

$$(\varphi_X)_{*1} \circ (\iota_1)_{*1} = (\varphi_X)_{*1} \circ (\pi_X)_{*1} \circ (\iota_2 \circ h)_{*1} \quad (\text{e 6.389})$$

$$= (\psi_2)_{*1} \circ (\iota_2 \circ h)_{*1}. \quad (\text{e 6.390})$$

But

$$\psi_2 \circ \iota_2 = 0. \quad (\text{e 6.391})$$

Therefore

$$(\varphi_X \circ \iota_1)_{*1} = 0. \quad (\text{e 6.392})$$

□

Lemma 6.6. *Let A be a unital infinite dimensional simple C^* -algebra with real rank zero, stable rank one and with unperforated $K_0(A)$, let $x \in A$ be a normal element and let $\varphi_X : C(X) \rightarrow A$ be the unital monomorphism induced by x , where $X = \text{sp}(x)$. Suppose that $y \in A$ is another normal element such that $\text{sp}(y) = Y$ and $[\lambda - x] = [\lambda - y]$ in $K_1(A)$ for all $\lambda \notin X \cup Y$ and suppose that $X \cap Y \neq \emptyset$.*

Suppose also that functions $f_1, f_2, \dots, f_n \in C(X \cap Y)$ are n mutually orthogonal projections with $1_{C(X \cap Y)} = \sum_{i=1}^n f_i$. Then, for any non-zero projection $e \in A$ and any n mutually orthogonal nonzero projections such that $\sum_{i=1}^n e_i = e$, there is a normal element $x_1 \in eAe$ with $\text{sp}(x_1) = X \cap Y$ satisfying the following: for any normal elements $x_0, y_0 \in (1 - e)A(1 - e)$ with finite spectrum in X and Y , respectively,

$$f_i(x_1) = e_i, \quad i = 1, 2, \dots, n, \quad (\text{e 6.393})$$

$$(\psi_1)_{*1} = (\varphi_X)_{*1} \quad \text{and} \quad (\text{e 6.394})$$

$$(\psi_2)_{*1} = (\varphi_Y)_{*1}, \quad (\text{e 6.395})$$

where $\psi_1 : C(X) \rightarrow A$ and $\psi_2 : C(Y) \rightarrow A$ are defined by $\psi_1(f) = f(x_0 + x_1)$ for all $f \in C(X)$ and $\psi_2(f) = f(y_0 + x_1)$ for all $f \in C(Y)$.

Proof. Let $\pi_X : C(X \cup Y) \rightarrow C(X)$, $\pi_Y : C(X \cup Y) \rightarrow C(Y)$, $\pi_{X \cap Y}^X : C(X) \rightarrow C(X \cap Y)$ and $\pi_{X \cap Y}^Y : C(Y) \rightarrow C(X \cap Y)$ be quotient maps. By the Universal Coefficient Theorem, there are $\kappa_1 \in KK(C(X), A)$ such that $\kappa_1|_{K_1(C(X))} = -(\varphi_X)_{*1}$ and $\kappa_1|_{K_0(C(X))} = 0$ and $\kappa_2 \in KK(C(Y), A)$ such that $\kappa_2|_{K_1(C(Y))} = (\varphi_Y)_{*1}$ and $\kappa_2|_{K_0(C(Y))} = 0$. Consider the pull back:

$$\begin{array}{ccc} C(X \cup Y) & \xrightarrow{\pi_X} & C(X) \\ \downarrow \pi_Y & & \downarrow \pi_{X \cap Y}^X \\ C(Y) & \xrightarrow{\pi_{X \cap Y}^Y} & C(X \cap Y). \end{array} \quad (\text{e 6.396})$$

By a Mayer-Vietoris Theorem (see, for example, 21.5.1 of [1]), one has the following six-term exact sequence:

$$\begin{array}{ccccccc} KK(C(X \cap Y), A) & \xrightarrow{(-[\varphi_{X \cap Y}^X], [\varphi_{X \cap Y}^Y])} & KK(C(X), A) \oplus KK(C(Y), A) & \xrightarrow{[\varphi_X] + [\varphi_Y]} & KK(C(X \cup Y), A) \\ \uparrow & & & & \downarrow \\ KK^1(C(X \cup Y), A) & \xleftarrow{[\varphi_X] + [\varphi_Y]} & KK^1(C(X), A) \oplus KK^1(C(Y), A) & \xleftarrow{(-[\varphi_{X \cap Y}^X], [\varphi_{X \cap Y}^Y])} & KK^1(C(X \cap Y), A). \end{array}$$

By the assumption and the proof of 6.5,

$$(\varphi_X)_{*1} \circ (\pi_X)_{*1} = (\varphi_X \circ \pi_X)_{*1} = (\varphi_Y \circ \pi_Y)_{*1} = (\varphi_Y)_{*1} \circ (\pi_Y)_{*1}.$$

It follows that

$$([\varphi_{X \cap Y}^X] + [\varphi_{X \cap Y}^Y])(\kappa_1, \kappa_2) = 0. \quad (\text{e 6.397})$$

The exactness of the Mayer-Vietoris sequence above shows that there is $\kappa_3 \in KK(C(X \cap Y), A)$ such that

$$(-[\varphi_X], [\varphi_Y])(\kappa_3) = (\kappa_1, \kappa_2), \quad (\text{e 6.398})$$

or

$$-[\varphi_{X \cap Y}^X](\kappa_3) = \kappa_1 \quad \text{and} \quad [\varphi_{X \cap Y}^Y](\kappa_3) = \kappa_2. \quad (\text{e 6.399})$$

Let $\kappa_3|_{K_1(C(X \cap Y))} = \lambda$ be as an element in $\text{Hom}(K_1(C(X \cap Y)), K_1(A))$. Then (e 6.399) implies that

$$\lambda \circ (\varphi_{X \cap Y}^X)_{*1} = (\varphi_X)_{*1} \text{ and } \lambda \circ (\varphi_{X \cap Y}^Y)_{*1} = (\varphi_Y)_{*1} \quad (\text{e 6.400})$$

It follows from 5.3 that there is a unital monomorphism $\psi'_1 : C(X \cap Y) \rightarrow eAe$ such that

$$(\psi'_1)_{*1} = \lambda \text{ and } \psi'_1(f_i) = e_i, \quad i = 1, 2, \dots, n. \quad (\text{e 6.401})$$

Let $z_{X \cap Y} \in C(X \cap Y)$ be the identity function on $X \cap Y$. Choose $x_1 \in eAe$ such that $x_1 = \psi'_1(z_{X \cap Y})$. Choose any normal element $x_0 \in B \subset eAe$, where B is a finite dimensional C^* -subalgebra of eAe with $1_B = e$ such that $\text{sp}(x_0) \subset X$. Define $\psi'' : C(X) \rightarrow (1 - e)A(1 - e)$ by $\psi''(f) = f(x_1)$ for all $f \in C(X)$ and define $\psi_1 : C(X) \rightarrow A$ by

$$\psi_1(f) = f(x_0 + x_1).$$

Then, since $x_0 \in B$, $\psi''(f) \in B$ for all $f \in B$. It follows that $(\psi'')_{*1} = 0$. Therefore, by (e 6.400),

$$(\psi_1)_{*1} = \lambda \circ (\pi_{X \cap Y}^X)_{*1} = (\varphi_X)_{*1}. \quad (\text{e 6.402})$$

Define $\psi_2 : C(Y) \rightarrow A$ by $\psi_2(g) = g(x_1 + y_0)$ for all $g \in C(Y)$ and for any normal element $y_0 \in B$ with $\text{sp}(y_0) \subset Y$. We also have

$$(\psi_2)_{*1} = \lambda \circ (\pi_{X \cap Y}^Y)_{*1} = (\varphi_Y)_{*1}. \quad (\text{e 6.403})$$

□

Let A be a unital simple C^* -algebra with $T(A) \neq \emptyset$. Let $\varphi_X : C(X) \rightarrow A$ be a unital monomorphism. Denote by $\varphi_X^\# : C(X)_{s.a.} \rightarrow \text{Aff}(T(A))$ the unital affine continuous map induced by φ . If $s : C(X) \rightarrow \mathbb{C}$ is a state of $C(X)$, denote by μ_s the probability Borel measure induced by s .

Theorem 6.7. *Let A be a unital separable simple C^* -algebra with real rank zero, stable rank one and with weakly unperforated $K_0(A)$. Let $x, y \in A$ be two normal elements with $\text{sp}(x) = X$ and $\text{sp}(y) = Y$. Denote $Z = X \cup Y$. Suppose that $[\lambda - x] = [\lambda - y]$ in $K_1(A)$ for all $\lambda \notin Z$.*

(1) *Then*

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq D_c^e(x, y). \quad (\text{e 6.404})$$

(2) *Moreover, if the pair (x, y) has a hub at $X \cap Y$, then*

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq D_c(x, y). \quad (\text{e 6.405})$$

Proof. Denote by $\varphi_X : C(X) \rightarrow A$ the unital monomorphism defined by $\varphi_X(f) = f(x)$ for all $f \in C(X)$ and $\varphi_Y : C(Y) \rightarrow A$ defined by $\varphi_Y(f) = f(y)$ for all $f \in C(Y)$. Let $\epsilon > 0$. Let $\mathcal{V}_1 \subset K_1(C(X))$ (in place of \mathcal{V}) be a finite subset, $\mathcal{P}_1 \subset K_0(C(X))$ (in place of \mathcal{P}) be a finite subset, $\mathcal{H}_1 \subset C(X)_{s.a.}$ (in place of \mathcal{H}) be a finite subset and let $\sigma_1 > 0$ (in place of σ) be required by 5.6 for $\epsilon/16$ and x .

Let $\mathcal{V}_2 \subset K_1(C(Y))$ (in place of \mathcal{V}) be a finite subset, $\mathcal{P}_2 \subset K_0(C(Y))$ (in place of \mathcal{P}) be a finite subset, $\mathcal{H}_2 \subset C(Y)_{s.a.}$ (in place of \mathcal{H}) be a finite subset and let $\sigma_2 > 0$ (in place of σ) be required by 5.6 for $\epsilon/16$ and y .

Without loss of generality, we may assume that \mathcal{H}_1 and \mathcal{H}_2 are in the unit balls of $C(X)$ and $C(Y)$, respectively. Moreover, we may assume that

$$\mathcal{P}_1 = \{f_1, f_2, \dots, f_{m_1}\} \text{ and } \mathcal{P}_2 = \{g_1, g_2, \dots, g_{m_2}\}, \quad (\text{e 6.406})$$

where $f_i \in C(X)$ and $g_i \in C(Y)$ which are projections such that

$$1_{C(X)} = \sum_{i=1}^{m_1} f_i \text{ and } 1_{C(Y)} = \sum_{i=1}^{m_2} g_i. \quad (\text{e 6.407})$$

Denote by F_1, F_2, \dots, F_{m_1} the clopen sets of X corresponding to projections f_1, f_2, \dots, f_{m_1} , and by G_1, G_2, \dots, G_{m_2} the clopen subsets of Y corresponding to projections g_1, g_2, \dots, g_{m_2} .

Let $X \cap Y = \sqcup_{j=1}^I S_j$, where S_1, S_2, \dots, S_I are distinct $\epsilon/8$ -connected components of $X \cap Y$. In particular, $\text{dist}(S_I, S_j) \geq \epsilon/8$. If $X \cap Y = \emptyset$, this notation simply means $I = 0$ (see 6.5).

Let $\eta > 0$ be such that $\eta < \epsilon$. By applying 6.3, there are mutually orthogonal projections $e_1, e_2, \dots, e_n \in A$ with $\sum_{i=1}^n e_i = 1_A$, $\lambda_1, \lambda_2, \dots, \lambda_n \in X$ and $\mu_1, \mu_2, \dots, \mu_n \in Y$ such that

$$\max\{|\tau(g(x)) - \tau(g(x_1))| : g \in \mathcal{H}_1\} < \sigma_1/2 \text{ for all } \tau \in T(A), \quad (\text{e 6.408})$$

$$\max\{|\tau(g(y)) - \tau(g(y_1))| : g \in \mathcal{H}_2\} < \sigma_2/2 \text{ for all } \tau \in T(A), \quad (\text{e 6.409})$$

$$D_c(x, x_1) \leq D_c^e(x, x_1) < \eta/4, \quad (\text{e 6.410})$$

$$D_c(y, y_1) \leq D_c^e(y, y_1) < \eta/4, \quad (\text{e 6.411})$$

$$D_c^e(x_1, y_1) < D_c^e(x, y) + \eta/4 \text{ and } \|x_1 - y_1\| < D_c(x, y) + \eta/4 \quad (\text{e 6.412})$$

$$\max_{1 \leq i \leq n} |\lambda_i - \mu_i| < D_c(x, y) + \eta/4, \quad (\text{e 6.413})$$

where

$$x_1 = \sum_{i=1}^n \lambda_i e_i \text{ and } y_1 = \sum_{i=1}^n \mu_i e_i. \quad (\text{e 6.414})$$

By the proof of 6.3 (by choosing even smaller δ) and by 2.16, we may assume that

$$[f_i(x_1)] = [f_i(x)], \quad i = 1, 2, \dots, m_1 \text{ and} \quad (\text{e 6.415})$$

$$[g_j(y_1)] = [g_j(y)] \quad j = 1, 2, \dots, m_2 \quad (\text{e 6.416})$$

We first consider case (1). Since the case that $X \cap Y = \emptyset$ will be dealt with in case (2), we will assume that $X \cap Y \neq \emptyset$. By the second part of 6.3, we may also assume that,

$$x_1 = \sum_{i=1}^I \lambda_i e_i^{(0)} + x'_2, \quad y_2 = \sum_{i=1}^I \lambda_i e_i^{(0)} + y'_2, \quad (\text{e 6.417})$$

$$D_c(x'_2, y'_2) < D_c^e(x, y) + \epsilon/4, \quad \tau\left(\sum_{i=1}^I e_i^{(0)}\right) < \min\{\sigma_1/2, \sigma_2/2\} \quad (\text{e 6.418})$$

for all $\tau \in T(A)$, where $\{e_1^{(0)}, e_2^{(0)}, \dots, e_I^{(0)}\}$ is a set of mutually orthogonal non-zero projections, $\lambda_i \in S_i$, $i = 1, 2, \dots, I$, $x'_2, y'_2 \in (1 - p_0)A(1 - p_0)$ are normal elements with finite spectrum in X and Y , respectively, and where $p_0 = \sum_{i=1}^I e_i^{(0)}$.

Let $h_j = \chi_{S_j} \in C(X \cap Y)$, $j = 1, 2, \dots, I$.

By applying 6.6, there is a normal element $x_0 \in p_0 A p_0$ with $\text{sp}(x_0) = X \cap Y$ such that

$$(\psi_1)_* 1|_{\mathcal{V}_1} = (\varphi_X)_* 1|_{\mathcal{V}_1}, \quad (\text{e 6.419})$$

$$(\psi_2)_* 1|_{\mathcal{V}_2} = (\varphi_Y)_* 1|_{\mathcal{V}_2}, \quad (\text{e 6.420})$$

$$h_j(x_0) = e_j^{(0)}, \quad j = 1, 2, \dots, I, \quad (\text{e 6.421})$$

where $\psi_1 : C(X) \rightarrow p_0 A p_0$ is defined by $\psi_1(f) = f(x_0)$ for all $f \in C(X)$ and $\psi_2 : C(Y) \rightarrow p_0 A p_0$ is defined by $\psi_2(f) = f(x_0)$ for all $f \in C(Y)$. Now consider $x_3 = x_0 + x'_2$ and $y_3 = x_0 + y'_2$. Note

that $\text{sp}(x_3) \subset X$ and $\text{sp}(y_3) \subset Y$. Define $\psi_3 : C(X) \rightarrow A$ by $\psi_3(f) = f(x_3)$ for all $f \in C(X)$ and $\psi_4 : C(Y) \rightarrow A$ by $\psi_4(f) = f(y_3)$ for all $f \in C(Y)$. Since x'_2 and y'_2 have finite spectra, by (e 6.419) and (e 6.420), we have

$$(\varphi_X)_*1|_{\mathcal{V}_1} = (\psi_3)_*1|_{\mathcal{V}_1} \quad \text{and} \quad (\varphi_Y)_*1|_{\mathcal{V}_2} = (\psi_4)_*1|_{\mathcal{V}_2}. \quad (\text{e 6.422})$$

For each i , if $F_i \cap Y = \emptyset$, i.e., $F_i \cap G_k = \emptyset$ for all k , we compute that

$$[\psi_3(f_i)] = [f_i(x_3)] = [f_i(x'_2)] = [f_i(x_1)] = [f_i(x)] \quad (\text{e 6.423})$$

$i = 1, 2, \dots, I$. If $F_i \cap Y \neq \emptyset$, let H_i be the subset of $\{j : j = 1, 2, \dots, I\}$ such that $h_j \leq f_i$. We then have

$$[\psi_3(f_i)] = \sum_{j \in H_i} [e_j^{(0)}] + [f_i(x'_2)] = [f_i(x_1)] = [f_i(x)]. \quad (\text{e 6.424})$$

Similarly, if $G_i \cap X = \emptyset$,

$$[\psi_4(g_i)] = [g_i(y_3)] = [g_i(y'_2)] = [g_i(y_1)] = [g_i(y)]. \quad (\text{e 6.425})$$

If $G_i \cap X \neq \emptyset$, let H'_i be the subset of $\{j : j = 1, 2, \dots, I\}$ such that $h_j \leq g_i$. We then have

$$[\psi_4(g_i)] = \sum_{j \in H'_i} [e_j^{(0)}] + [g_i(y'_2)] = [g_i(y_1)] = [g_i(y)]. \quad (\text{e 6.426})$$

In other words,

$$(\psi_3)_*0|_{\mathcal{P}_1} = (\varphi_X)_*0|_{\mathcal{P}_1} \quad \text{and} \quad (\psi_4)_*0|_{\mathcal{P}_2} = (\varphi_Y)_*0|_{\mathcal{P}_2}. \quad (\text{e 6.427})$$

By applying 5.6, using (e 6.422), (e 6.427), (e 6.408), (e 6.409) and (e 6.418), we obtain a unitary $u_1, u_2 \in A$ such that

$$\|u_1^* x u_1 - x_3\| < \epsilon/16 \quad \text{and} \quad \|u_2^* y u_2 - y_3\| < \epsilon/16. \quad (\text{e 6.428})$$

By (e 6.418) and by 3.5 there is a unitary $u_3 \in (1 - p_0)A(1 - p_0)$ such that

$$\|u_3^* x'_2 u_3 - y'_2\| < D_c^e(x, y) + \epsilon/16. \quad (\text{e 6.429})$$

Put $u_4 = p_0 + u_3$. Then

$$\|u_4^* x_3 u_4 - y_3\| = \|(x_0 + u_3^* x'_2 u_3) - (x_0 + y'_2)\| = \|u_3^* x'_2 u_3 - y'_2\| \quad (\text{e 6.430})$$

$$< D_c^e(x, y) + \epsilon/16. \quad (\text{e 6.431})$$

Therefore

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) < D_c^e(x, y) + 5\epsilon/8. \quad (\text{e 6.432})$$

This proves the case (1).

Now we turn to case (2).

If $X \cap Y = \emptyset$, by the assumption that $[\lambda - x]$ and $[\lambda - y]$ are the same in $K_1(A)$ for all $\lambda \notin X \cup Y$, $\lambda - x \in \text{Inv}_0(A)$ for all $\lambda \notin X$ and $\lambda - y \in \text{Inv}_0(A)$ for all $\lambda \notin Y$. Thus this special case has been proved in 3.6.

Thus we will then assume again $X \cap Y \neq \emptyset$. Some of the argument above will be repeated. In this case, we may assume that, if $F_j \cap G_k \neq \emptyset$, there are at least one i such that $\lambda_i, \mu_i \in F_j \cap G_k$.

Note that, in this case, $F_j \cap G_k$ is a non-empty clopen subset of $X \cap Y$. In fact $X \cap Y$ is a disjoint union of those $F_j \cap G_k$. Call them T_1, T_2, \dots, T_k . Then $k \leq n$. We may assume that $\{r(1), r(2), \dots, r(k)\} \subset \{1, 2, \dots, n\}$ such that $\lambda_{r(j)}, \mu_{r(j)} \in T_j$, $j = 1, 2, \dots, k$.

Since A is simple and infinite dimensional, one can find a non-zero projection $e_{r(j)}^{(0)} \leq e_{r(j)}$ such that $[e_{r(j)}^{(0)}] = [e_{r(1)}^{(0)}]$ in $K_0(A)$, $j = 1, 2, \dots, k$, and

$$\tau\left(\sum_{j=1}^k e_{r(j)}^{(0)}\right) < \min\{\sigma_1/2, \sigma_2/2\} \text{ for all } \tau \in T(A). \quad (\text{e 6.433})$$

Let $p_0 = \sum_{j=1}^k e_{r(j)}^{(0)}$, $p = 1_A - p_0$, $p_i = e_i - e_i^{(0)}$, if $i \in \{r(1), r(2), \dots, r(k)\}$ and $p_i = e_i$, if $i \notin \{r(1), r(2), \dots, r(k)\}$. Put

$$x_2 = \sum_{i=1}^n \lambda_i p_i \text{ and } y_2 = \sum_{i=1}^n \mu_i p_i. \quad (\text{e 6.434})$$

Note, by (e 6.413), that

$$\|x_2 - y_2\| \leq \max_{1 \leq i \leq n} |\lambda_i - \mu_i| < D_c(x, y) + \eta/4. \quad (\text{e 6.435})$$

Let $h'_j = \chi_{T_j}$, $j = 1, 2, \dots, k$.

By applying 6.6, there is a normal element $x_0 \in p_0 A p_0$ with $\text{sp}(x_0) = X \cap Y$ such that

$$(\psi_1)_* 1|_{\mathcal{V}_1} = (\varphi_X)_* 1|_{\mathcal{V}_1}, \quad (\text{e 6.436})$$

$$(\psi_2)_* 1|_{\mathcal{V}_2} = (\varphi_Y)_* 1|_{\mathcal{V}_2}, \quad (\text{e 6.437})$$

$$h'_i(x_0) = e_{r(i)}^{(0)}, \quad i = 1, 2, \dots, k, \quad (\text{e 6.438})$$

where $\psi_1 : C(X) \rightarrow p_0 A p_0$ is defined by $\psi_1(f) = f(x_0)$ for all $f \in C(X)$ and $\psi_2 : C(Y) \rightarrow p_0 A p_0$ is defined by $\psi_2(f) = f(x_0)$ for all $f \in C(Y)$. Now consider $x_3 = x_0 + x_2$ and $y_3 = x_0 + y_2$. Note that $\text{sp}(x_3) \subset X$ and $\text{sp}(y_3) \subset Y$. Define $\psi_3 : C(X) \rightarrow A$ by $\psi_3(f) = f(x_3)$ for all $f \in C(X)$ and $\psi_4 : C(Y) \rightarrow A$ by $\psi_4(f) = f(y_3)$ for all $f \in C(Y)$. Since x_2 and y_2 have finite spectra, by (e 6.419), we have

$$(\varphi_X)_* 1|_{\mathcal{V}_1} = (\psi_3)_* 1|_{\mathcal{V}_1} \text{ and } (\varphi_Y)_* 1|_{\mathcal{V}_2} = (\psi_4)_* 1|_{\mathcal{V}_2} \quad (\text{e 6.439})$$

For each i , if $F_i \cap Y = \emptyset$, i.e., $i \notin \{r(1), r(2), \dots, r(k)\}$, we compute that

$$[\psi_3(f_i)] = [\psi_1(f_i)] = [\varphi_X(f_i)]. \quad (\text{e 6.440})$$

If $F_i \cap Y \neq \emptyset$, we also have

$$[\psi_3(f_i)] = \sum_{h'_j \leq f_i} [e_{r(j)}^{(0)}] + \left(\sum_{h'_j \leq f_i} [e_{r(j)} - e_{r(j)}^{(0)}] + \sum_{\lambda_j \in F_i, j \neq r(j)} [e_i] \right) \quad (\text{e 6.441})$$

$$= \left[\sum_{\lambda_i \in F_i} e_j \right] = [\varphi_X(f_i)], \quad (\text{e 6.442})$$

$j = 1, 2, \dots, k$. In other words,

$$(\psi_3)_* 0|_{\mathcal{P}_1} = (\varphi_X)_* 0|_{\mathcal{P}_1}. \quad (\text{e 6.443})$$

By applying 5.6, using (e 6.443), (e 6.439) and (e 6.408) and (e 6.433), we obtain a unitary such that

$$\|u^*xu - x_3\| < \epsilon/16. \quad (\text{e 6.444})$$

On the other hand, for each j , $\psi_2(g_j) = \sum_{h'_i \leq g_j} h'_i(x_0)$.

It follows that

$$[\psi_2(g_j)] = \sum_{\mu_{r(i)} \in G_j} [e_{r(i)}^{(0)}]. \quad (\text{e 6.445})$$

Therefore,

$$[\psi_4(g_j)] = [(\psi_2)(g_j)] + [g_j(y_2)] \quad (\text{e 6.446})$$

$$= \sum_{\mu_{r(i)} \in G_j} [e_{r(i)}^{(0)}] + \sum_{\mu_{r(j)} \in G_j} [e_{r(i)} - e_{r(i)}^{(0)}] + \sum_{\mu_s \in G_j, s \notin \{r(i): 1 \leq i \leq r(k)\}} [e_s] \quad (\text{e 6.447})$$

$$= \sum_{\mu_s \in G_j} [e_s] = [\varphi_Y(g_j)]. \quad (\text{e 6.448})$$

It follows that

$$(\psi_4)_*0|_{\mathcal{P}_2} = (\varphi_Y)_*0|_{\mathcal{P}_2}. \quad (\text{e 6.449})$$

It follows from (e 6.449), (e 6.439), (e 6.409), (e 6.433) and 5.6 that there is a unitary $v \in A$ such that

$$\|v^*yv - y_3\| < \epsilon/16. \quad (\text{e 6.450})$$

We also have (using (e 6.435))

$$\|x_3 - y_3\| = \|(x_0 + x_2) - (x_0 + y_2)\| = \|x_2 - y_2\| < D_c(x, y) + \epsilon/4. \quad (\text{e 6.451})$$

It follows that

$$\|u^*xu - v^*yv\| < \epsilon/16 + D_c(x, y) + \epsilon/4 + \epsilon/16 < D_c(x, y) + \epsilon/2. \quad (\text{e 6.452})$$

Therefore

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) < D_c(x, y) + \epsilon/2 \quad (\text{e 6.453})$$

for all $\epsilon > 0$. The theorem follows. \square

Remark 6.8. It is probably helpful to be reminded that

$$D_c^e(x, y) \leq \min\{D^T(x, y), 2D_c(x, y)\}.$$

and if both X and Y are connected, $D_c^e(x, y) = D_c(x, y)$.

Corollary 6.9. *Let A be a unital separable simple infinite dimensional C^* -algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$ and let $x \in A$ be a normal element with $\text{sp}(x) = X$. Then, for any $\epsilon > 0$, any $\sigma > 0$ and any finite subset $\{\xi_1, \xi_2, \dots, \xi_k\} \subset \text{sp}(x)$, there*

is a set of mutually orthogonal non-zero projections $\{e_1, e_2, \dots, e_k\}$ of A and a normal element $x_0 \in (1-p)A(1-p)$ with $\text{sp}(x_0) = X$ such that

$$\|x - (x_0 + \sum_{i=1}^k \xi_i e_i)\| < \epsilon, \quad (\text{e 6.454})$$

$$\tau(\sum_{i=1}^k e_i) < \sigma \text{ for all } \tau \in T(A), \quad (\text{e 6.455})$$

$$(\varphi_1)_{*1} = (\varphi_2)_{*1}, \quad (\text{e 6.456})$$

where $p = \sum_{i=1}^k e_i$, $\varphi_1, \varphi_2 : C(X) \rightarrow A$ is defined by $\varphi_1(f) = f(x)$ and $\varphi_2(f) = f(x_0) + \sum_{i=1}^k f(\xi_i) e_i$ for all $f \in C(X)$.

Proof. This is merely a refinement of that of 5.4. The issue is that we now insist that $\text{sp}(x_0) = X$. The proof is contained in the proof of the case (1) in the proof of 6.7 by letting $X = Y$. With the notation in the proof of the case (1) in 6.7, we have $x_3 = x_0 + x'_2$, where x'_2 has finite spectrum but $\text{sp}(x'_2)$ can be $\epsilon/16$ -dense in X . Note that $\text{sp}(x_0) = X$. We may assume, without loss of generality, that $x'_2 = \sum_{i=1}^k \xi_i p_i + x''_2$, where $\{p_1, p_2, \dots, p_k\}$ is a set of mutually orthogonal non-zero projections and where x''_2 is a normal element in $(1 - \sum_{i=1}^k p_i)A(1 - \sum_{i=1}^k p_i)$ with $\text{sp}(x''_2) \subset X$. Since A is simple and has the property (SP), there are non-zero projections $e'_i \leq p_i$, $i = 1, 2, \dots, k$, such that

$$\sum_{i=1}^k \tau(e'_i) < \sigma \text{ for all } \tau \in T(A). \quad (\text{e 6.457})$$

We still have, as in (e 6.428), a unitary $u_1 \in A$ such that

$$\|u_1^* x u_1 - x_3\| < \epsilon/16 \quad (\text{e 6.458})$$

Then we have

$$\|x - (u_1(x_0 + x''_2)u_1^* + \sum_{i=1}^k \xi_i u_1 e'_i u_1^*)\| < \epsilon/16. \quad (\text{e 6.459})$$

Choose the new x_0 to be $u_1(x_0 + x''_2)u_1^*$ and e_i to be $u_1 e'_i u_1^*$.

□

Corollary 6.10. *Let A be a unital separable simple C^* -algebra of real rank zero, stable rank one and weakly unperforated $K_0(A)$, let $x, y \in A$ be two normal elements with $\text{sp}(x) = X$ and $\text{sp}(y) = Y$. Then the pair (x, y) has hub at $X \cap Y$, if one of the following holds:*

- (1) $X = Y$ is connected;
- (2) $X \cap Y$ is connected and it contains an open ball with radius $D_c(x, y)$;
- (3) for every connected component S of X , either $S = X \cap Y$ or $\text{dist}(\xi, X \cap Y) > D_c(x, y)$ for all $\xi \in S$;
- (4) $X \cap Y = \emptyset$.

Consequently,

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq D_c(x, y),$$

if one of the above conditions holds.

Proof. It is clear that (1) and (4) follow from the definition immediately. It is also clear that (3) holds since $X \cap Y$ must be connected and no point in $X \cap Y$ can pair with any point outside $X \cap Y$ with a distance no more than $D_c(x, y)$.

To see (2), let $X = \sqcup_{i=1}^{m_1} F_i$ and $Y = \sqcup_{j=1}^{m_2} G_j$, where $\{F_1, F_2, \dots, F_{m_1}\}$ and $\{G_1, G_2, \dots, G_{m_2}\}$ are of mutually disjoint clopen sets. Since $X \cap Y$ is connected, we may assume that $X \cap Y = F_1 \cap G_1$, and $F_i \cap G_j = \emptyset$, if $(i, j) \neq (1, 1)$.

Let

$$d = \min\{\text{dist}(G_1, G_j) : j = 2, 3, \dots, m_2\} > 0.$$

Choose $\epsilon_0 = d/16$. Let $0 < \epsilon < \epsilon_0$. Suppose that $F = \{\lambda_1, \lambda_2, \dots, \lambda_K\} \subset X$ and $G = \{\mu_1, \mu_2, \dots, \mu_L\} \subset Y$ are finite subsets, $x_1 = \sum_{i=1}^K \lambda_i e_i$ and $y_1 = \sum_{j=1}^L \mu_j p_j$, where $\{e_1, e_2, \dots, e_K\}$ and $\{p_1, p_2, \dots, p_L\}$ are two sets of mutually orthogonal projections in A such that $\sum_{i=1}^K e_i = 1_A = \sum_{j=1}^L p_j$, and such that

$$D_c(x_1, x) < \epsilon \text{ and } D_c(y_1, y) < \epsilon. \quad (\text{e 6.460})$$

Therefore

$$D_c(x, y) - 2\epsilon < D_c(x_1, y_1) < D_c(x, y) + 2\epsilon. \quad (\text{e 6.461})$$

Then there is $\lambda_i \in F_1 \cap G_1 = X \cap Y$ such that

$$\text{dist}(\lambda_i, G_j) > (D_c(x, y) - \epsilon) + d > D_c(x_1, y_1) + d - 3\epsilon > D_c(x, y).$$

for all $j \neq 1$. In other words there is j such that $\mu_j \in F_1 \cap G_1$ and $(i, j) \in R_{x_1, y_2}$ (see 6.1). Therefore the pair (x, y) has a hub at $X \cap Y$. □

Corollary 6.11. *Let A be a unital separable simple C^* -algebra of real rank zero, stable rank one and weakly unperforated $K_0(A)$, let $x, y \in A$ be two normal elements with $\text{sp}(x) = X$ and $\text{sp}(y) = Y$. If X or Y is connected and if $[\lambda - x] = [\lambda - y]$ in $K_1(A)$ for all $\lambda \notin X \cup Y$, then*

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq D_c(x, y).$$

Proof. If X or Y is connected, by 4.10, $D_c(x, y) = D_c^e(x, y)$. The corollary follows from 6.7. □

7 Distance between unitary orbits of normal elements with different K_1 maps

In this section we will show that, without the condition that $[\lambda - x] = [\lambda - y]$ in $K_1(A)$ for $\lambda \notin X \cup Y$ in the statement of 6.7, $D_c(x, y)$ alone may have little to do with the distance of the unitary orbits of x and y as 7.2 shows. However, 7.3 provides us some description of the upper as well as lower bound for the distance between unitary orbits of normal elements.

Let A be a unital C^* -algebra and let $x, y \in A$ be two normal elements. Let $X = \text{sp}(x)$ and $Y = \text{sp}(y)$. Denote by $d_H(X, Y)$ the Hausdorff distance between the subset X and Y . Define

$$\rho(x, y) = \max\{d_H(X, Y), \rho_1(x, y)\}, \quad (\text{e 7.462})$$

where

$$\rho_1(x, y) = \sup\{\text{dist}(\lambda, X) + \text{dist}(\lambda, Y) : \lambda \notin X \cup Y, (\lambda - x)(\lambda - y)^{-1} \notin \text{Inv}_0(A)\}. \quad (\text{e 7.463})$$

Let

$$\rho_x(x, y) = \sup\{\text{dist}(\lambda, X) : \lambda \notin X \cup Y, (\lambda - x)^{-1}(\lambda - y) \notin \text{Inv}_0(A)\} \text{ and } \quad (\text{e 7.464})$$

$$\rho_y(x, y) = \sup\{\text{dist}(\lambda, Y) : \lambda \notin X \cup Y, (\lambda - x)^{-1}(\lambda - y) \notin \text{Inv}_0(A)\}. \quad (\text{e 7.465})$$

The following is a result of Ken Davidson ([5]). The proof is exact the same as that in [5].

Proposition 7.1. *Let A be a unital C^* -algebra and let $x, y \in A$ be two normal elements. Then*

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \geq \rho(x, y). \quad (\text{e 7.466})$$

Theorem 7.2. *Let A be a unital, infinite dimensional, separable simple C^* -algebra with real rank zero, stable rank one, weakly unperforated $K_0(A)$ and $K_1(A) \neq \{0\}$. Then*

(1) *for any unitary $u_1 \in A$ with $\text{sp}(u_1) = \mathbb{T}$ (the unit circle), there is a unitary $u_2 \in A$ such that $[u_1] \neq [u_2]$ in $K_1(A)$,*

$$D_c(u_1, u_2) = 0 \text{ and } \text{dist}(\mathcal{U}(u_1), \mathcal{U}(u_2)) = 2; \quad (\text{e 7.467})$$

(2) *For any compact subset $X \subset \mathbb{C}$ such that $\mathbb{C} \setminus X$ is not connected and for any normal element $x \in A$ with $\text{sp}(x) = X$, there exists a normal element $y \in A$ such that*

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \quad (\text{e 7.468})$$

$$\geq 2 \sup\{\text{dist}(\lambda, \text{sp}(x)) : \lambda \text{ in bounded components of } \mathbb{C} \setminus \text{sp}(x)\} \quad (\text{e 7.469})$$

$$\text{and } D_c(x, y) = 0. \quad (\text{e 7.470})$$

Proof. It is clear that (1) follows from (2). So we will prove (2). As in the beginning of the proof of 5.3, there is a unital simple AH-algebra B with slow dimension growth and with real rank zero such that

$$(K_0(B), K_0(B)_+, [1_B], K_1(B)) = (K_0(A), K_0(A)_+, [1_A], K_1(A))$$

and since $\rho_B(K_0(B))$ and $\rho_A(K_0(A))$ are dense in $\text{Aff}(T(B))$ and $\text{Aff}(T(A))$, respectively, the above also gives an affine homeomorphism from $\text{Aff}(T(B))$ to $\text{Aff}(T(A))$ which is compatible with the above identification. We will use this fact in the proof of (2).

Let $x \in A$ be a normal element with $\text{sp}(x) = X$. Put

$$d = \sup\{\text{dist}(\lambda, \text{sp}(x)) : \lambda \text{ in bounded components of } \mathbb{C} \setminus \text{sp}(x)\}.$$

Let S be the union of all bounded components of $\mathbb{C} \setminus \text{sp}(x)$. Then

$$\sup\{|\lambda| : \lambda \in S\} \leq \|x\|.$$

In particular, $d \leq \|x\|$. There is $\lambda_0 \in S$ such that $\text{dist}(\lambda_0, \text{sp}(x)) = d_0 > 0$. So $d \geq d_0$. The set

$$S_1 = \{\xi \in S : \|\xi\| \geq \text{dist}(\xi, \text{sp}(x)) \geq d_0\} \quad (\text{e 7.471})$$

is compact. It follows that there is $\lambda \in S_1$ such that

$$\text{dist}(\lambda, \text{sp}(x)) = d. \quad (\text{e 7.472})$$

Let $\varphi_1 : C(X) \rightarrow A$ be the unital monomorphism defined by $\varphi(f) = f(x)$ for all $f \in C(X)$. By [22], since $K_1(A) \neq \{0\}$, there exists a unital monomorphism $\psi : C(X) \rightarrow B \subset A$ such that $\psi_{*0} = \varphi_{*0}$, $\tau \circ \psi = \tau \circ \varphi$ for all $\tau \in T(B) = T(A)$ and $[\lambda - \psi(z)] \neq [\lambda - x]$ in $K_1(A)$, where $z : X \rightarrow X$ is the identity function.

Let $y \in \mathcal{U}(\psi(z))$. It follows from 4.6 and 2.21 that

$$D_c(x, y) = 0. \quad (\text{e 7.473})$$

Since $(\lambda - x)^{-1}(\lambda - \psi(z)) \notin \text{Inv}_0(A)$, by 7.1,

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \geq 2\text{dist}(\lambda, \text{sp}(x)) = 2d. \quad (\text{e 7.474})$$

□

Please note that the above 7.2 does not follow from the following theorem.

Theorem 7.3. *Let A be a unital separable simple C^* -algebra with real rank zero, stable rank one and weakly unperforated $K_0(A)$ and let $x, y \in A$ be two normal elements.*

Then

$$\rho(x, y) \leq \text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq \min\{D_1, D_2\}, \quad (\text{e 7.475})$$

where

$$D_1 = \max\{D^T(x, y), \max\{\rho_x(x, y), \rho_y(x, y)\}\} + \min\{\rho_x(x, y), \rho_y(x, y)\}, \quad (\text{e 7.476})$$

$$D_2 = D_c^e(x, y) + 2\min\{\rho_x(x, y), \rho_y(x, y)\}. \quad (\text{e 7.477})$$

Proof. Let $X = \text{sp}(x)$ and $Y = \text{sp}(y)$. Let $d = D^T(x, y)$, $d/2 > \epsilon > 0$ and let

$$S = \{\lambda \in \mathbb{C} : \lambda \notin X \cup Y, (\lambda - x)^{-1}(\lambda - y) \notin \text{Inv}_0(A)\}. \quad (\text{e 7.478})$$

Note that the closure \bar{S} of S is compact. Let $\xi_1, \xi_2, \dots, \xi_{L'}$ be a finite subset of $X \cup Y \cup \bar{S}$ such that it is $\epsilon/32$ -dense in $X \cup Y \cup \bar{S}$. Let N_1, N_2, \dots, N_L be all possible finite unions of $O(\xi_i, \epsilon/32)$'s such that $N_j \cap Y \neq Y$ for $j = 1, 2, \dots, L$. For each i , let

$$\eta_i = \inf\{d_\tau(f_{(N_i)_{d+\epsilon/32}}(x)) - d_\tau(f_{N_i}(y)) : \tau \in T(A)\}. \quad (\text{e 7.479})$$

It follows from 2.20 that $\eta_i > 0$, $i = 1, 2, \dots, L$. Choose

$$0 < \eta < \min\{\epsilon/4, \min\{\eta_i : 1 \leq i \leq L\}/8\}. \quad (\text{e 7.480})$$

Let $\delta > 0$ with $\delta < \min\{\epsilon/2^{10}, \eta/16\}$. Let $g_i \in C(X \cup Y \cup \bar{S})$ be such that $0 \leq g_i(t) \leq 1$, $g_i(t) = 1$ if $t \in (N_i)_{d+\epsilon/8}$, $g_i(t) = 0$ if $t \notin (N_i)_{d+\epsilon/4}$, $i = 1, 2, \dots, L$.

There are distinct points $\zeta_1, \zeta_2, \dots, \zeta_K \in \bar{S}$ such that

$$\cup_{i=1}^K O(\zeta_i, \delta/4) \supset \bar{S}. \quad (\text{e 7.481})$$

Let S_1, S_2, \dots, S_K be compact subsets of \bar{S} such that

$$\zeta_i \in S_i \text{ and } \text{diam}(S_i) < \delta/2, \quad i = 1, 2, \dots, K. \quad (\text{e 7.482})$$

Since $\text{sp}(x)$ is compact, there are $\lambda_1, \lambda_2, \dots, \lambda_K \in \text{sp}(x)$ such that

$$\text{dist}(\lambda_i, \zeta_i) = \text{dist}(\text{sp}(x), \zeta_i), \quad (\text{e 7.483})$$

$i = 1, 2, \dots, K$.

Let $\mathcal{H} = \{z, g_i : 1 \leq i \leq L\}$, where z represents the identity function on $X \cup Y \cup \bar{S}$.

By 6.9 and 4.11, there are nonzero mutually orthogonal projections $\{e_1, e_2, \dots, e_K, e_{K+1}, \dots, e_k\}$, a unitary w in A and normal elements $x_0 \in (1 - Q_1)A(1 - Q_1)$ with $\text{sp}(x_0) = X$ and $y_0 \in (1 - Q_2)A(1 - Q_2)$ with $\text{sp}(y_0) = Y$ satisfy the following:

$$\|f(x) - (f(x_0) + \sum_{i=1}^k f(\lambda_i)e_i)\| < \delta/4 \text{ for all } f \in \mathcal{H}, \quad (\text{e 7.484})$$

$$\|f(y) - w^*(f(y_0) + \sum_{i=K+1}^k f(\lambda_i)e_i)w\| < \delta/4 \text{ for all } f \in \mathcal{H} \quad (\text{e 7.485})$$

$$[\lambda - x] = [\lambda - x_1] \text{ for all } \lambda \notin X \text{ and} \quad (\text{e 7.486})$$

$$\tau(\sum_{i=1}^k e_i) < \eta/2, \quad (\text{e 7.487})$$

$$D_c(x_0 + \sum_{i=1}^K \lambda_i e_i, y_0) < D_c^e(x, y) + \delta/4 \quad (\text{e 7.488})$$

for all $\tau \in T(A)$, where $Q_1 = \sum_{i=1}^k e_i$ and $Q_2 = \sum_{i=K+1}^k e_i$, $x_1 = x_0 + \sum_{i=1}^k \lambda_i e_i$. and $\{\lambda_{K+1}, \lambda_{K+2}, \dots, \lambda_k\}$ is $\delta/4$ -dense in $X \cap Y$. As in the proof of 5.3, there are normal elements $h_i \in e_i A e_i$ such that $\text{sp}(h_i) = S_i$, $\lambda - h_i \in \text{Inv}_0(e_i A e_i)$ for all $\lambda \notin S_i$, $i = 1, 2, \dots, K$.

Define

$$x_2 = x_0 + \sum_{i=1}^K h_i + \sum_{i=K+1}^k \lambda_i e_i \quad (\text{e 7.489})$$

It follows that

$$\|x - x_2\| \leq \|x - x_1\| + \|x_1 - x_2\| \quad (\text{e 7.490})$$

$$< \delta/4 + \left\| \sum_{i=1}^K \lambda_i e_i - \sum_{i=1}^K h_i \right\| \quad (\text{e 7.491})$$

$$\leq \delta/4 + \max\{\|\lambda_i e_i - h_i\| : 1 \leq i \leq K\} \quad (\text{e 7.492})$$

$$< \delta/4 + \rho_x(x, y). \quad (\text{e 7.493})$$

Let $Z = X \cup \bar{S}$. Define $\psi : C(\Omega) \rightarrow A$ by $\psi(f) = f(x_2)$ for all $f \in C(\Omega)$ and define $\psi_Y : C(\Omega) \rightarrow A$ by $\psi_Y(g) = g(y)$ for $g \in C(\Omega)$. Let $\lambda \notin Z \cup Y = X \cup Y \cup \bar{S}$. By the assumption and (e 7.485),

$$[\lambda - x_2] = [\lambda - y] \text{ for all } \lambda \notin Z \cup Y. \quad (\text{e 7.494})$$

Since $\delta < \eta/16$, by (e 7.484),

$$\tau(f_i(x_1)) > \tau(f_i(x)) - \eta/16 \text{ for all } \tau \in T(A), \quad (\text{e 7.495})$$

$i = 1, 2, \dots, L$. Let

$$d_1 = d_H(\text{sp}(x_2), Y) = \max\{d_H(X, Y), \rho_y(x, y)\}. \quad (\text{e 7.496})$$

Let $O \subset Y \cup X \cup \bar{B}$ be an open subset with $O \cap Y \neq Y$. If $O_{\epsilon/2} \cap Y = Y$, since A is simple,

$$d_\tau(\psi_Y(f_O)) < d_\tau(\psi_Y(f_{O_{\epsilon/2}})) \text{ for all } \tau \in T(A). \quad (\text{e 7.497})$$

But we also have that $O_{d_1+\epsilon} \cap Z \supset \text{sp}(x_2)$. Then $\psi(f_{O_{d_1+\epsilon}}) = 1_A$. It follows that

$$d_\tau(\psi_Y(f_O)) < d_\tau(\psi_Y(f_{O_{\epsilon/2}})) = d_\tau(\psi(f_{O_{d_1+\epsilon}})). \quad (\text{e 7.498})$$

If $O_{\epsilon/2} \cap Y \neq Y$, let $O_{\epsilon/32} \cap \{\xi_1, \xi_2, \dots, \xi_{L'}\} = \xi_{k_1}, \xi_{k_2}, \dots, \xi_{k_l}$. Then $O \subset \cup_{j=1}^l O(\xi_{k_j}, \epsilon/32) \subset O_{\epsilon/16}$. It follows that there is j such that

$$O \subset N_j \subset (N_j)_{d+\epsilon/16} \subset O_{d+\epsilon/8}. \quad (\text{e 7.499})$$

By (e 7.487), (e 7.495), (e 7.499) and (e 7.479), we have

$$d_\tau(\psi(f_{O_{d+\epsilon}})) - d_\tau(\psi_Y(f_O)) > d_\tau(f_{O_{d+\epsilon}}(x_0)) - d_\tau(\psi_Y(f_{N_j})) - \eta/2 \quad (\text{e 7.500})$$

$$\geq \tau(f_{O_{d+\epsilon}}(x_1)) - d_\tau(\psi_Y(f_{N_j})) - \eta/2 - \eta/2 \quad (\text{e 7.501})$$

$$\geq \tau(g_j(x_1)) - d_\tau(\psi_Y(f_{N_j})) - \delta/4 - \eta \quad (\text{e 7.502})$$

$$> \tau(g_j(x)) - d_\tau(\psi_Y(f_{N_j})) - \eta/16 - \delta/4 - \eta \quad (\text{e 7.503})$$

$$\geq d_\tau(f_{(N_j)_{d+\epsilon/16}}(x)) - d_\tau(\psi_Y(f_{N_j})) - 17\eta/16 - \delta/4 \quad (\text{e 7.504})$$

$$\geq \eta_j - 17\eta/16 - \eta/64 > 0 \quad (\text{e 7.505})$$

for all $\tau \in T(A)$. By (e 7.500)-(e 7.505) and (e 7.496)

$$D^T(x_2, y) \leq \max\{D^T(x, y) + \epsilon, \rho_y(x, y)\}. \quad (\text{e 7.506})$$

It follows from this, (e 7.494), 4.9 and 6.7 that

$$\text{dist}(\mathcal{U}(x_2), \mathcal{U}(y)) \leq \max\{D^T(x, y) + \epsilon, \rho_y(x, y)\}. \quad (\text{e 7.507})$$

Combining this with (e 7.490) and (e 7.493), we have ($\delta < \epsilon/2^{10}$)

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq \max\{D^T(x, y) + \epsilon, \rho_y(x, y)\} + \rho_x(x, y) + \epsilon/2^{12} \quad (\text{e 7.508})$$

for all $\epsilon > 0$. Therefore

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq \max\{D^T(x, y), \rho_y(x, y)\} + \rho_x(x, y). \quad (\text{e 7.509})$$

Since we may switch the position of x and y , we conclude that

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq \max\{D^T(x, y), \max\{\rho_x(x, y), \rho_y(x, y)\}\} + \min\{\rho_x(x, y), \rho_y(x, y)\}. \quad (\text{e 7.510})$$

On the hand, we have

$$D_c^e(x_2, y) \leq D_c(x_0 + \sum_{i=1}^K h_i, y_0) \quad (\text{e 7.511})$$

$$\leq \rho_x(x, y) + D_c(x_0 + \lambda_i e_i, y_0) \leq \rho_x(x, y) + D_c^e(x, y) + \delta/4. \quad (\text{e 7.512})$$

It follows from 6.7 that

$$\text{dist}(\mathcal{U}(x_2), \mathcal{U}(y)) \leq \rho_x(x, y) + D_c^e(x, y) + \delta/4. \quad (\text{e 7.513})$$

By (e 7.493),

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq D_c^e(x, y) + 2\rho_x(x, y) + \delta/4. \quad (\text{e 7.514})$$

Since we can exchange x with y in the above proof, finally, we conclude that

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq D_c^e(x, y) + 2 \min\{\rho_x(x, y), \rho_y(x, y)\}. \quad (\text{e 7.515})$$

□

In some special case below exact formula for distance can be stated.

Corollary 7.4. *Let A be a unital separable simple C^* -algebra of real rank zero, stable rank one and weakly unperforated $K_0(A)$ and let $x, y \in A$ be two normal elements. If $D_c(x, y) = 0$, then*

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) = \rho_1(x, y). \quad (\text{e 7.516})$$

If $X = Y$ and X is connected, then

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq \max\{D_c(x, y), (1/2)\rho_1(x, y)\} + (1/2)\rho_1(x, y). \quad (\text{e 7.517})$$

Proof. In this case $\text{sp}(x) = \text{sp}(y)$ and $d_H(X, Y) = 0$. Therefore $\rho_x(x, y) = \rho_y(x, y)$ and

$$\rho_1(x, y) = \rho_x(x, y) + \rho_y(x, y) = 2\rho_x(x, y).$$

In case that X is connected, by 2.21, $D_c(x, y) = D^T(x, y)$. Thus the corollary follows from 7.3. \square

8 Lower bound

Last section gives both upper bound and lower bound for the distance between unitary orbits of normal elements. However, the lower bound are all given by the bounded components of $\mathbb{C} \setminus X \cup Y$ which give different K_1 -information of the corresponding normal elements. In this section, we will discuss the lower bound for distance between unitary orbits of normal elements who have the same K_1 -information outside of $X \cup Y$.

Theorem 8.1. *There exists a constant $C > 0$ satisfying the following: Let A be a unital separable AF -algebra and let $x, y \in A$ be two normal elements. Then*

$$C \cdot D_c(x, y) \leq \text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq D_c(x, y). \quad (\text{e 8.518})$$

Proof. Let $C = c^{-1}$ be in the statement of Theorem 4.2 of [6]. Without loss of generality, we may assume that $\|x\|, \|y\| \leq 1$. It follows from 3.6 that it suffices to show that

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \geq C \cdot D_c(x, y). \quad (\text{e 8.519})$$

We will show that

$$\|x - y\| \geq C \cdot D_c(x, y), \quad (\text{e 8.520})$$

Put $d = D_c(x, y)$. Let $\epsilon > 0$. It follows from [17] that there are $\lambda_1, \lambda_2, \dots, \lambda_n \in \text{sp}(x)$, $\mu_1, \mu_2, \dots, \mu_m \in \text{sp}(y)$, two sets of mutually orthogonal non-zero projections $\{p_1, p_2, \dots, p_n\}$ and $\{q_1, q_2, \dots, q_m\}$ in A such that

$$\|x - \sum_{i=1}^n \lambda_i p_i\| < \epsilon/16 \quad \text{and} \quad \|y - \sum_{j=1}^m \mu_j q_j\| < \epsilon/16. \quad (\text{e 8.521})$$

Put $x_1 = \sum_{i=1}^n \lambda_i p_i$ and $y_1 = \sum_{j=1}^m \mu_j q_j$. Without loss of generality, by the virtue of 2.17, we may also assume that

$$D_c(x, x_1) < \epsilon/16 \quad \text{and} \quad D_c(y, y_1) < \epsilon/16. \quad (\text{e 8.522})$$

Since $D_c(\cdot, \cdot)$ is a metric,

$$D_c(x_1, y_1) \geq D_c(x, y) - \epsilon/8. \quad (\text{e 8.523})$$

Let $\epsilon > \delta > 0$ be given. Since A is an AF-algebra, there is a finite dimensional C^* -algebra $B \subset A$ such that there are mutually orthogonal projections $\{p'_1, p'_2, \dots, p'_n\}$ and $\{q'_1, q'_2, \dots, q'_m\}$ in B such that

$$\|p_i - p'_i\| < \delta/16n \text{ and } \|q_j - q'_j\| < \delta/16m, \quad (\text{e 8.524})$$

$i = 1, 2, \dots, n$ and $j = 1, 2, \dots, m$. Put $x_2 = \sum_{i=1}^n \lambda_i p'_i$ and $y_2 = \sum_{j=1}^m \mu_j q'_j$. Therefore, by the virtue of 2.17 and choosing sufficiently small δ ,

$$\|x_1 - x_2\| < \epsilon/16, \quad \|y_1 - y_2\| < \epsilon/16 \text{ and} \quad (\text{e 8.525})$$

$$D_c(x_1, x_2) < \epsilon/16 \text{ and } D_c(y_1, y_2) < \epsilon/16. \quad (\text{e 8.526})$$

It follows from (e 8.523) and (e 8.525) that

$$D_c(x_2, y_2) \geq d - \epsilon/4. \quad (\text{e 8.527})$$

This has to hold in B too. By Theorem 4.2 of [6],

$$\text{dist}(x_2, y_2) \geq C(d - \epsilon/4). \quad (\text{e 8.528})$$

It follows that

$$\|x - y\| \geq C(d - \epsilon/4) - \epsilon/4 = C \cdot d - C\epsilon/4 - \epsilon/4 \quad (\text{e 8.529})$$

for any $\epsilon > 0$.

So we get (e 8.520). For any unitary u , by (e 8.520), we have

$$\|u^*xu - y\| \geq C \cdot D_c(u^*xu, y) = C \cdot D_c(x, y).$$

That implies (e 8.519) holds. \square

Lemma 8.2. *Let A be a unital simple C^* -algebra with $TR(A) = 0$, let $x, y \in A$ be two normal elements. Let $\eta > 0$. Suppose that*

$$\tau(f(x)) > \tau(g(y)) \quad (\text{e 8.530})$$

for some positive functions $f \in C(\overline{X_\eta})$ and $g \in C(\overline{Y_\eta})$ and for some $\tau \in T(A)$. Then, for any $\epsilon > 0$, there is a projection $p \in A$, and a finite dimensional C^ -subalgebra $B \subset A$ with $1_B = p$, and normal elements $x_0, y_0 \in (1 - p)A(1 - p)$, $x_1, y_1 \in B$ such that $\text{sp}(x_1) \subset \overline{X_\eta}$, $\text{sp}(y_1) \subset \overline{Y_\eta}$ with*

$$\|x - (x_0 + x_1)\| < \epsilon, \quad \|y - (y_0 + y_1)\| < \epsilon \quad (\text{e 8.531})$$

$$t_0(f(x_1)) > t_0(g(y_1)) \quad (\text{e 8.532})$$

for some $t_0 \in T(B)$.

Proof. Let $\tau(f(x)) > \tau(g(y))$ for some $\tau \in T(A)$ and let $d = \tau(f(x)) - \tau(g(y)) > 0$. Fix a separable C^* -subalgebra $C \subset A$ such that $x, y \in C$. Since A has tracial rank zero, there exists a sequence of projections $\{p_n\}$ and a sequence of finite dimensional C^* -subalgebras $\{B_n\}$ such that $1_{B_n} = p_n$, $n = 1, 2, \dots$, such that

$$\lim_{n \rightarrow \infty} \|p_n c - c p_n\| = 0 \text{ for all } c \in C; \quad (\text{e 8.533})$$

$$\lim_{n \rightarrow \infty} \text{dist}(p_n c p_n, B_n) = 0 \text{ and} \quad (\text{e 8.534})$$

$$\lim_{n \rightarrow \infty} \max\{t(1 - p_n) : t \in T(A)\} = 0. \quad (\text{e 8.535})$$

It follows from [16] and [11] that there exists normal elements $x_n^{(0)}, y_n^{(0)} \in (1 - p_n)A(1 - p_n)$, $x_n^{(1)}, y_n^{(1)} \in B_n$ such that

$$\lim_{n \rightarrow \infty} \|x - (x_n^{(0)} + x_n^{(1)})\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|y - (y_n^{(0)} + y_n^{(1)})\| = 0. \quad (\text{e 8.536})$$

Therefore, we may assume that $\text{sp}(x_n^{(0)}), \text{sp}(x_n^{(1)}) \subset \overline{X_\eta}$ and $\text{sp}(y_n^{(0)}), \text{sp}(y_n^{(1)}) \subset \overline{Y_\eta}$. It follows that

$$\lim_{n \rightarrow \infty} \|f(x) - (f(x_n^{(0)}) + f(x_n^{(1)}))\| = 0 \text{ and } \lim_{n \rightarrow \infty} \|g(y) - (g(y_n^{(0)}) + g(y_n^{(1)}))\| = 0. \quad (\text{e 8.537})$$

By (e 8.535), we may assume that

$$t(1 - p_n) < d/4 \text{ for all } t \in T(A). \quad (\text{e 8.538})$$

It follows, for all sufficiently large n , that

$$\tau(f(x_n^{(1)})) > \tau(f(x)) - d/4 - d/4 > \tau(g(y_n^{(1)})) \quad (\text{e 8.539})$$

Note B_n is a finite direct sum of simple C^* -algebras. If, for all tracial states $t \in T(B_n)$,

$$t(f(x_n^{(1)})) \leq t(g(y_n^{(1)})), \quad (\text{e 8.540})$$

then, by [3], there is a sequence $\{z_{k,n}\} \subset B_n$ such that

$$\sum_{k=1}^{\infty} z_{k,n}^* z_{k,n} = f(x_n^{(1)}) \text{ and } \sum_{k=1}^{\infty} z_{k,n} z_{k,n}^* \leq g(y_n^{(1)}). \quad (\text{e 8.541})$$

Since $B_n \subset A$, this would imply that

$$\tau(f(x_n^{(1)})) \leq \tau(g(y_n^{(1)})) \text{ for all } \tau \in T(A) \quad (\text{e 8.542})$$

which contradicts with (e 8.539). □

Theorem 8.3. *There is a constant $C > 0$ satisfying the following: Let A be a unital separable simple C^* -algebra with $TR(A) = 0$ and let $x, y \in A$ be two normal elements. Then*

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \geq C \cdot D_T(x, y). \quad (\text{e 8.543})$$

If $[\lambda - x] = [\lambda - y]$ in $K_1(A)$ for all $\lambda \notin \text{sp}(x) \cup \text{sp}(y)$, then

$$D_c^e(x, y) \geq \text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \geq C \cdot D_T(x, y). \quad (\text{e 8.544})$$

Proof. Note that the second part of the theorem follows from the first part and 6.7. So we will only prove the first part of the theorem.

Let C be the constant c^{-1} in the statement of Theorem 4.2 of [6]. Let $r = D_T(x, y)$ and let $1/2 > \epsilon > 0$. Suppose that $K > 0$ such that $\|x\|, \|y\| \leq K$ and D is a closed ball with the center at the origin and radius larger than K . Denote by $\varphi_X, \varphi_Y : C(D) \rightarrow A$ the unital homomorphisms defined by $\varphi_X(f) = f(x)$ and $\varphi_Y(f) = f(y)$ for all $f \in C(D)$. We will show that

$$\|x - y\| \geq Cr. \quad (\text{e 8.545})$$

There is an open subset O of D such that

$$d_\tau(\varphi_X(f_O)) > d_\tau(\varphi_Y(f_{O_{s_1}})), \quad (\text{e 8.546})$$

for some $\tau \in T(A)$, where $s_1 = r - \epsilon/8$. It follows that

$$d_\tau(\varphi_X(f_O)) > \tau(\varphi_Y(g)), \quad (\text{e 8.547})$$

where $g \in C(D)$ such that $0 \leq g \leq 1$, $g(\xi) = 1$ if $\text{dist}(\xi, O) < r - \epsilon/2$ and $g(\xi) = 0$ if $\text{dist}(\xi, O) \geq r - \epsilon/4$. There is $\delta > 0$ such that

$$\tau(f_\delta(\varphi_X(f_O))) > \tau(\varphi_Y(g)). \quad (\text{e 8.548})$$

By 8.2, there is a projection $p \in A$, a finite dimensional C^* -algebra $B \subset A$ with $1_B = p$, normal elements $x_0, y_0 \in (1-p)A(1-p)$, $x_1, y_1 \in B$ with $\text{sp}(x_0), \text{sp}(y_0), \text{sp}(x_1), \text{sp}(y_1) \subset D$ such that

$$\|x - (x_0 + x_1)\| < \epsilon/16, \quad \|y - (y_0 + y_1)\| < \epsilon/16 \quad \text{and} \quad (\text{e 8.549})$$

$$t_0(f_\delta(f_O(x_1))) > t_0(g(y_1)) \quad (\text{e 8.550})$$

for some $t_0 \in B$. Therefore

$$d_{t_0}(f_O(x_1)) \geq t_0(f_\delta(f_O(x_1))) > d_{t_0}(f_{O_{r-\epsilon/2}}(y_1)). \quad (\text{e 8.551})$$

It follows that, in B ,

$$D_c(x_1, y_1) \geq r - \epsilon/2. \quad (\text{e 8.552})$$

It follows from Theorem 2.4 of [6] that

$$\|x_1 - y_1\| \geq C(r - \epsilon/2). \quad (\text{e 8.553})$$

It follows from (e 8.549) that

$$\|x - y\| \geq \|(x_0 + x_1) - (y_0 + y_1)\| - \|x - (x_0 + x_1)\| - \|(y_0 + y_1) - y\| \quad (\text{e 8.554})$$

$$\geq \|x_1 - y_1\| - \epsilon/8 \geq Cr - C\epsilon/2 - \epsilon/8. \quad (\text{e 8.555})$$

Hence

$$\|x - y\| \geq C \cdot D_T(x, y). \quad (\text{e 8.556})$$

□

Lemma 8.4. *Let A be a unital simple C^* -algebra with $TR(A) = 0$. Suppose that $p, q \in A$ are two non-zero projections such that*

$$\tau(p) > \tau(q) \quad (\text{e 8.557})$$

for some $\tau \in T(A)$. Then

$$\|(1-q)p\| = 1 \quad (\text{e 8.558})$$

Proof. We first apply 8.2. Let $x = p$, $y = q$ and $f = g$ be identity function on $[0, 1]$. Then, for any $\epsilon > 0$, there are a non-zero projection $e \in A$ and a finite dimensional C^* -algebra $B \subset A$ with $1_B = e$, non-zero projections $p_1, q_1 \in B$ and $p_0, q_0 \in (1 - e)A(1 - e)$ such that

$$\|p - (p_0 + p_1)\| < \epsilon/4, \quad \|q - (q_0 + q_1)\| < \epsilon/4 \quad \text{and} \quad t_0(p_1) > t_0(q_1) \quad (\text{e 8.559})$$

for some $t_0 \in T(B)$. Since B is finite dimensional, we write $B = M_{n_1} \oplus M_{n_2} \oplus \cdots \oplus M_{n_k}$. Accordingly, we may write

$$p_1 = (p_{1,1}, \dots, p_{1,n_k}), \quad q_1 = (q_{1,1}, \dots, q_{1,n_k}), \quad (\text{e 8.560})$$

$$(\text{e 8.561})$$

where $p_{1,i}, q_{1,i} \in M_{n_i}$, $i = 1, 2, \dots, n_k$. The last condition in (e 8.559) implies, for some i ,

$$\text{rank} p_{1,i} > \text{rank} q_{1,i}. \quad (\text{e 8.562})$$

Let M_{n_i} act on H_i ($\dim H_i = n_i$). Then, by counting the rank, $(1_{B_i} - q_{1,i})H_i \cap p_{1,i}H_i \neq \{0\}$. Therefore

$$\|(1_{B_i} - q_{1,i})p_{1,i}\| = 1. \quad (\text{e 8.563})$$

It follows that

$$\|(e - q_1)p_1\| = 1 \quad (\text{e 8.564})$$

Let $\xi \in (1_{B_i} - q_{1,i})H_i \cap p_{1,i}H_i$ be a unit vector. Define a projection $e_0 \in B(H_i) = M_{n_i}$ by $e_0(x) = \langle x, \xi \rangle \xi$ for all $x \in H_i$. Then $e_0 \in M_{n_i} \subset A$ is a non-zero projection. Moreover,

$$e_0 \leq e - q_1 \quad \text{and} \quad e_0 \leq p_1. \quad (\text{e 8.565})$$

Therefore

$$\|(1 - q)p\| \geq \|1 - (q_0 + q_1)(p_0 + p_1)\| - \epsilon/2 \quad (\text{e 8.566})$$

$$\geq \|e(1 - (q_0 + q_1)(p_0 + p_1))e\| - \epsilon/2 = \|(e - q_1)p_1\| - \epsilon/2 \quad (\text{e 8.567})$$

$$= 1 - \epsilon/2. \quad (\text{e 8.568})$$

It follows that (e 8.558) holds. □

Theorem 8.5. *Let A be a separable simple C^* -algebra with $TR(A) = 0$ and let $x, y \in A$ be two normal elements. Then*

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \geq d_T(x, y). \quad (\text{e 8.569})$$

If A is a finite dimensional C^ -algebra then*

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \geq d_c(x, y). \quad (\text{e 8.570})$$

Proof. Let $0 < d < d_T(x, y)$. Note that in a finite dimensional C^* -algebra $d_c(x, y) = d_T(x, y)$. Let $\epsilon > 0$. We assume that $\epsilon < \frac{d_T(x, y) - d}{4}$. We also assume that $\|x\|, \|y\| > 2\epsilon$. Denote $X = \text{sp}(x)$ and $Y = \text{sp}(y)$. Choose any pair of $x' \in \mathcal{U}(x)$ and $y' \in \mathcal{U}(y)$. Since $\text{sp}(x') = X$, $\text{sp}(y') = Y$, $d_c(x', y') = d_c(x, y)$, and $d_T(x', y') = d_T(x, y)$, to simplify the notation, without loss of generality, it suffices to show that $\|x - y\| \geq d_T(x, y)$.

By the assumption, there is an open disc $O = O(\lambda, \eta)$ such that

$$d_\tau(f_O(x)) > d_\tau(f_{O_{d+\epsilon}}(y)) \quad (\text{e 8.571})$$

for some $\tau \in T(A)$ (including the case that A is finite dimensional).

Let e_1 be the spectrum projection of x corresponding to open set $O_{\eta+\epsilon}$ and e_2 be the spectrum projection of y corresponding to O_d in A^{**} .

Denote by $z = y(1 - e_2) = (1 - e_2)y$. Then

$$\text{sp}_{(1-e_2)A^{**}(1-e_2)}(z) = Y \setminus O_d \cap Y \quad (\text{e 8.572})$$

(as an element in $(1 - e_2)A^{**}(1 - e_2)$). The inequality (e 8.571) implies that $Y \neq O_d \cap Y$. In particular,

$$\text{dist}(\lambda, \text{sp}_{(1-e_2)A^{**}(1-e_2)}(z)) \geq d. \quad (\text{e 8.573})$$

We also note that

$$\|xe_1 - \lambda e_1\| < \eta + \epsilon. \quad (\text{e 8.574})$$

Therefore

$$\|(1 - e_2)(y - \lambda)e_1\| \leq \|(1 - e_2)(y - x)e_1\| + \|(1 - e_2)(x - \lambda)e_1\| \quad (\text{e 8.575})$$

$$< \|(1 - e_2)(y - x)e_1\| + \eta + \epsilon. \quad (\text{e 8.576})$$

It follows that

$$\|(1 - e_2)(y - x)e_1\| > \|(1 - e_2)(y - \lambda)e_1\| - \eta - \epsilon. \quad (\text{e 8.577})$$

One has

$$(1 - e_2)(y - \lambda)e_1 = (y - \lambda)(1 - e_2)e_1 \quad (\text{e 8.578})$$

$$= (1 - e_2)(y - \lambda)(1 - e_2)e_1 \quad (\text{e 8.579})$$

$$= (z - \lambda)(1 - e_2)e_1. \quad (\text{e 8.580})$$

Let z_1 be the inverse of $z - \lambda$ in $(1 - e_2)A^{**}(1 - e_2)$. Then

$$\|(1 - e_2)e_1\| \leq \|z_1(z - \lambda)(1 - e_2)e_1\| \leq \|z_1\| \|(z - \lambda)(1 - e_2)e_1\|. \quad (\text{e 8.581})$$

It follows that

$$\|(z - \lambda)(1 - e_2)e_1\| \geq \frac{\|(1 - e_2)e_1\|}{\|z_1\|} \quad (\text{e 8.582})$$

$$= \text{dist}(\lambda, \text{sp}_{(1-e_2)A^{**}(1-e_2)}(z)) \|(1 - e_2)e_1\| \quad (\text{e 8.583})$$

$$\geq (\eta + d) \|(1 - e_2)e_1\|. \quad (\text{e 8.584})$$

By (e 8.577) and (e 8.578), one concludes that

$$\|y - x\| \geq \|(1 - e_2)(y - x)e_1\| > \|(1 - e_2)(y - \lambda)e_1\| - \eta - \epsilon \quad (\text{e 8.585})$$

$$\geq \|(z - \lambda)(1 - e_2)e_1\| - \eta - \epsilon \quad (\text{e 8.586})$$

$$\geq (\eta + d) \|(1 - e_2)e_1\| - \eta - \epsilon. \quad (\text{e 8.587})$$

If A is finite dimensional, then $e_1, e_2 \in A$. By (e 8.571),

$$\text{ran} e_1 > \text{ran} e_2. \quad (\text{e 8.588})$$

As in the proof of 8.4, this implies that $\|(1 - e_2)e_1\| = 1$. From this and from (e 8.585) to (e 8.587),

$$\|x - y\| \geq d. \quad (\text{e 8.589})$$

It follows that

$$\|x - y\| \geq d_c(x, y). \quad (\text{e 8.590})$$

Let e_0 be the spectral projection of x corresponding to the closed set $\{\xi \in \mathbb{C} : \text{dist}(\xi, \lambda) \leq \eta\}$ and e_3 be the spectral projection of y corresponding to the open subset $O_{d+\epsilon}$ in A^{**} . Note that e_0 is a closed projection and e_3 is an open projection. If A is a simple infinite dimensional C^* -algebra with $TR(A) = 0$, by [2], there are projections $p_1, q_1 \in A$ such that

$$e_0 \leq q_1 \leq e_1 \text{ and } e_2 \leq p_1 \leq e_3. \quad (\text{e 8.591})$$

By (e 8.571),

$$\tau(q_1) > \tau(p_1). \quad (\text{e 8.592})$$

It follows from 8.4 that

$$\|(1 - p_1)q_1\| = 1. \quad (\text{e 8.593})$$

By (e 8.585),

$$\begin{aligned} \|x - y\| &\geq (\eta + d)\|(1 - e_2)e_1\| - \eta - \epsilon & (\text{e 8.594}) \\ &\geq (\eta + d)\|(1 - p_1)(1 - e_2)e_1q_1\| - \eta - \epsilon & (\text{e 8.595}) \\ &= (\eta + d)\|(1 - p_1)q_1\| - \eta - \epsilon & (\text{e 8.596}) \\ &\geq d - \epsilon. & (\text{e 8.597}) \end{aligned}$$

The theorem follows. \square

Remark 8.6. Suppose that x, y are normal elements in a separable simple C^* -algebra A with $TR(A) = 0$, and $\text{sp}(x)$ is a subset of a straight line L_1 and $\text{sp}(y)$ is a subset of another straight line. If L_1 and L_2 are parallel, by applying 8.5, one can show that $d_c(x, y) = D_c(x, y)$. Hence by 8.5 and 3.6,

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) = D_c(x, y) = d_c(x, y).$$

If L_1 and L_2 are perpendicular, one can also show $\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) = D_c(x, y)$. However, in 2.25 of section 2, there are x, y in finite dimensional C^* -algebra, with $x = x^*, y = iy^*$ and

$$d_c(x, y) = 1 < \sqrt{2} = D_c(x, y),$$

So we get an example such that

$$d_c(x, y) < \text{dist}(\mathcal{U}(x), \mathcal{U}(y)) = D_c(x, y).$$

Corollary 8.7. Let A be a unital AF -algebra and let $x, y \in A$ be two normal elements. Suppose that $D_c(x, y) = d_c(x, y)$. Then

$$\text{dist}(\mathcal{U}(x), \mathcal{U}(y)) = D_c(x, y). \quad (\text{e 8.598})$$

Corollary 8.8. *Let A be a unital simple separable C^* -algebra with $TR(A) = 0$ and let $x, y \in A$ be two normal elements with $\text{sp}(x) = X$ and $\text{sp}(y) = Y$. Then*

$$\max\{C \cdot D_T(x, y), d_T(x, y), \rho_1(x, y)\} \leq \text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq \min\{D_1, D_2\}, \quad (\text{e 8.599})$$

where

$$\begin{aligned} D_1 &= \max\{D^T(x, y), \max\{\rho_x(x, y), \rho_y(x, y)\}\} + \min\{\rho_x(x, y), \rho_y(x, y)\} \text{ and} \\ D_2 &= \max\{D_c^e(x, y) + \min\{\rho_x(x, y), \rho_y(x, y)\}\} + \min\{\rho_x(x, y), \rho_y(x, y)\}. \end{aligned} \quad (\text{e 8.600})$$

Suppose that

$$[\lambda - x] = [\lambda - y] \text{ in } K_1(A) \quad (\text{e 8.601})$$

for all $\lambda \notin X \cup Y$. Then

$$\max\{C \cdot D_T(x, y), d_T(x, y)\} \leq \text{dist}(\mathcal{U}(x), \mathcal{U}(y)) \leq D_c^e(x, y). \quad (\text{e 8.602})$$

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